Division of a distribution by a polynomial. The Mckibben method in $\mathbb{R}^2$

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25-26 June 2015 - Thessaloniki
Analysis Summer Meeting

Some preliminaries

Let V be a vector space over $\mathbb{C}$. A linear functional in V is a homeomorphism $u : V \to \mathbb{C}$. The linear forms (over V) become a vector space $\text{Hom}(V, \mathbb{C})$ by acting on functions $\phi \in V$ as follows:

(i) $\langle cu, \phi \rangle = c\langle u, \phi \rangle, \ \forall c \in \mathbb{C}, \ u \in \text{Hom}(V, \mathbb{C}), \ \phi \in V.$

(ii) $\langle u + u, \phi \rangle = \langle u, \phi \rangle + \langle v, \phi \rangle, \ \forall u, v \in \text{Hom}(V, \mathbb{C}), \ \phi \in V.$

Definition

Let $X \subseteq \mathbb{R}^n$. Set

$$C_c^\infty(X) \equiv \mathcal{D}(X) = \{\phi \in C^\infty(X) : supp(\phi) \subseteq X \text{ compact}\}.$$

The space $\mathcal{D}(X)$ of C^∞ functions with compact support is also called the space of test functions (in X).

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Topology on $\mathcal{D}(\mathbb{R}^n)$

Definition

We say that a sequence $\{\phi_n\}_n \subset \mathcal{D}(X)$ converges on $\phi \in \mathcal{D}(X)$ if the following conditions hold:

1. There exists a compact set $K \subset X$ which contains all the supports of ϕ_n, i.e.
$$\bigcup_k \text{supp}(\varphi_k) \subset K.$$
2. For every $\alpha \in \mathbb{N}_0^n$, the sequence of partial derivatives $\partial^\alpha \phi_n$ converges uniformly on $\partial^\alpha \phi$.

A linear functional u is called continuous if for every sequence $(\phi_n)_n$ in $\mathcal{D}(X)$ such that $\phi_n \to \phi \in \mathcal{D}(X)$ we have

$$\lim_n \langle u, \phi_n \rangle = \langle u, \lim_n \phi \rangle = \langle u, \phi \rangle. \quad (1)$$

Some preliminaries

example A function $f \in C(\mathbb{R}^n;\mathbb{C})$ defines a linear functional u_f as follows:

$$u_f : \mathcal{D}(\mathbb{R}^n) \to \mathbb{C} \quad \textit{with} \quad \phi \mapsto u_f(\phi) \equiv \langle f, \phi \rangle := \int_{\mathbb{R}^n} f(x)\phi(x)\,dx.$$

Definition

A $u \in \mathrm{Hom}(\mathcal{D}(X), \mathbb{C})$ is called a *distribution* if $\forall K \subset X$ compact, $\exists M \geq 0$ *and* $N \in \mathbb{Z}_+$ such that

$$|\langle u, \phi \rangle| \leq M \sum_{|\alpha| \leq N} \sup |\partial^\alpha \phi|, \tag{2}$$

$\forall \phi \in \mathcal{D}(X)$ with $\mathit{supp}(\phi) \subseteq K$. ■

We denote with $\mathcal{D}'(X)$ the set of all distributions in X.The set

$$\mathcal{S}(\mathbb{R}^n) = \{\phi : \mathbb{R}^n \to \mathbb{R} \ : \ \forall \alpha, \beta \in \mathbb{N}_0^n, \ \sup_{x \in \mathbb{R}^n} |x^\beta \partial^\alpha \phi(x)| < \infty\}.$$

is called Schwarz's space.

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The elements of $\mathcal{S}(\mathbb{R}^n)$ share the property that $\lim_{|x|\to\infty} |x^\beta \partial^\alpha \phi(x)| = 0$ and for this reason this set is called the space of rapidly decreasing functions.

Principal Values and Finite Parts of Integrals The function $f_\alpha : \mathbb{R} \to \mathbb{R}$ ($\alpha \in \mathbb{R}$) with $x \mapsto f_\alpha(x) := x^\alpha$ belongs to the space $L^1_{loc}(\mathbb{R})$ for $\alpha > -1$ and therefore it defines a distribution. For $\alpha = -1$, we define as the Cauchy Principal Value Integral and denote with $PV\left(\frac{1}{x}\right)$ the element of $Hom(\mathcal{D}(\mathbb{R});\mathbb{R})$

$$\phi \mapsto \langle PV\left(\frac{1}{x}\right), \phi\rangle := \lim_{\epsilon\to 0}\int_{\{|x|>\epsilon\}} \frac{\phi(x)}{x}dx.$$

The $PV\left(\frac{1}{x}\right)$ belongs to $\mathcal{D}'(\mathbb{R})$. For higher powers of x, we discard those boundary terms that tend to become infinite: we define the (Hadamard) Finite Part of $\frac{1}{x^n}$, ($n \in \mathbb{N},\ n \geq 2$)

$$\left\langle PV\left(\frac{1}{x^n}\right), \phi\right\rangle := \lim_{\epsilon\to 0}\int_{\{|x|>\epsilon\}} \frac{\phi(x) - \sum_{k=0}^{n-2}\phi^{(k)}(0)x^k}{x^n}dx. \qquad (3)$$

The division problem in $\mathbb{R}^n$

Let $U \subset \mathbb{R}^n$ open. Given a $u \in \mathcal{D}'(U)$ and a C^∞ function f in U, can we find a distribution v (in U) such that

$$u = fv;$$

Aim: give meaning (as a distribution) of the correspondence

$$\phi \mapsto \int \frac{\phi}{f} dx_1 \dots dx_n.$$

We want to determine an inverse (in the sense of distributions) to f, i.e. a distribution $v = \frac{u}{f} := FP\left(\frac{1}{f}\right)$ acting on $\mathcal{D}(U)$ by

$$\phi \mapsto \langle v, f\phi \rangle = \int \phi dx_1 \dots dx_n.$$

History:

Hormander proved that division is always possible [1958]. Moreover, (by applying Fourier transform), every LPD equation

$$P(\partial)u = f$$

with constant coefficients has a *tempered* solution u for every tempered f. In particular, there exists a tempered fundamental solution, i.e. $P(\partial)u = \delta$. [see Hor]

S. Lojasiewicz: The division is always possible even when f is real analytic.

J. Mckibben [1959] Explicit construction of an elementary (tempered) solution for $P(\partial)u = \delta$ in 2 dimensions.

M. F. Atiyah [1970] Applied Hironaka's Theorem of Resolution of Singularities to obtain another proof of the Theorems of Hormander and Lojasiewicz.

Peter Wagner [1998] Gives (tempered) fundamental solution for the operator

$$\partial_1^3 + \partial_2^3 + \partial_3^3 + 3a\partial_1\partial_2\partial_3, \quad a \in \mathbb{R} \setminus \{-1\}$$

by methods of elliptic integrals.

Division of a distribution by a polynomial in $\mathbb{R}$

For $v \in \mathcal{D}'(\mathbb{R})$ and an f a C^∞ function of one real variable, we seek for a $u \in \mathcal{D}'(\mathbb{R})$ such that

$$fu = v.$$

We seek for a finite part of $\frac{v}{f}$. If the set $f^{-1}(\{0\})$ consists of isolated points (the singularities) of finite order, the division problem is rather easy to be solved. If $f = P$ a polynomial in one real variable, with a partition of unity and a translation to the origin, the problem reduces to a local problem. Specifically, we can assume that $P(0) = 0$ and that the support of ϕ does not contain any other zero of P. Then, in a neighborhood of $x = 0$, we can write $P(x) = x^m \widetilde{P}(x)$, where $\widetilde{P}$ does not vanish for $x = 0$. Expanding ϕ as a Taylor sum of order m around the origin

$$\phi(x) = \sum_{j=0}^{m-1} \frac{x^j}{j!} \phi^{(j)}(0) + T_k(x),$$

where T_k is the remainder, we can define our distribution $FP\left(\frac{1}{x^m}\right)$ by

$$\left\langle FP\left(\frac{1}{x^m}\right),\phi\right\rangle = \int_0^\infty \frac{T_k(x)}{x^m}dx$$

(k is the least integer greater than $m-1$.) In other words, if $u \in \mathcal{D}'(\mathbb{R})$ such that $x^m u = \delta$ (in the sense of distributions), then

$$u = FP\left(\frac{1}{x^m}\right) + \sum_{k=0}^{m-1} c_k \partial_x^{(k)} \delta_0,$$

where $c_k \in \mathbb{C}$. Thus, we can perform division for any function $y(x)$ such that around $x = 0$

$$\frac{1}{y} = \sum_{i=-j}^{\infty} x^{\frac{i}{m}} \quad (m, j \in \mathbb{Z}_+). \tag{4}$$

Then we can define (for q a polynomial of degree m)

$$FP\int_{q(x,y)=0} \frac{\phi(x)}{y(x)}dx,$$

since along the branches of the curve $q(x, y) = 0$ we can represent $\frac{1}{y}$ as in (4).

Division of a distribution by a polynomial in $\mathbb{R}^2$

Theorem (Green's identity)

Let $m \in \mathbb{Z}_+$ and let $\Omega \subseteq \mathbb{R}^n$ open with smooth boundary. Let P be a PDO of order m. Then, if $w, u \in C^m(\Omega)$, the following identity holds:

$$\int_\Omega [w(x)P(u(x)) - u(x)P^* w(x)]dx = \int_{\partial\Omega} M(u, w) d\sigma(x),$$

where $d\sigma$ denotes the volume element on the boundary of Ω and $M(u, w)$ a function depending on u, w and their derivatives of order at most $m - 1$.

From now on, P_m will denote the principal part of P.

Write

$$M(u,w) = [N(x,a)w]u + H,$$

where a is the unit normal to $\partial\Omega$ at x, N is a PDO of order ≥ 1 homogenous and of order $m-1$ and H denote partial derivatives of order ≥ 1. Aim: Construction of an inverse of P^*. Choose v such that

$$\int_{\langle x,\xi\rangle - p \geq 0} vP^*\phi(x)dx = \int_{\langle x,\xi\rangle - p = 0} \phi(x)d\sigma,$$

where $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ with $|\xi|^2 = 1$, $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $p \in \mathbb{R}$. This is done by choosing $w = v$, where v solves $Pv = 0$ and that $M(v,\phi) = \phi$ on the hyperplane $\{\langle x,\xi\rangle = p\}$. Equivalently, v must satisfy

(i) $q(\partial_\xi)v = 0$

(ii) derivatives (with respect to x) of v of order $\leq m-2$ vanish on the hyperplane $\{\langle x,\xi\rangle - p = 0\}$

(iii) $\partial_\xi^{m-1}v = \frac{1}{q_m(\xi)}$ on the hyperplane $\{\langle x,\xi\rangle - p = 0\}$

Remark that:

(1) The above conditions that guarantee the existence of such a v are satisfied in some special cases. For example, when P_m has constant coefficients and the remaining coefficients are constant in a neighborhood of $x = 0$ and if $P_m(\xi) \neq 0$, for all values of $x, \ \xi, \ p$. In this case, a solution exists and is (locally around 0) analytic.

(2) Even for fixed values of $x, \ \xi, \ p$, and P_m has constant coefficients, v has a non-integrable singularity on a $\xi \in \Omega_\xi(0, 1)$ such that $L_m(\xi) = 0$.

The singularities of the polynomial

Write

$$q(\xi_1, \xi_2) = \sum_{i=0}^{m} q_i(\xi_1, \xi_2),$$

where q_i is a homogenous polynomial of degree i ($i \in \{0, 1, \dots, m\}$). The singularities of v lie on the variety

$$\{q_m b = 0\},$$

where b is the determinant of a matrix of a linear system of ODE's. Let $p_\xi(\lambda) := q(\lambda\xi)$, where $\lambda \in \mathbb{C}$. Suppose that $q_0 \neq 0$. Then p_ξ also irreducible. Let

$$S := \{(\xi, \lambda) \in \mathbb{C}^3 \ : \ \lambda = 0\}$$

Denote ρ the projection on the first coordinate on S :

$$\rho(\xi, \lambda) = \xi.$$

The multiplicity of a point $\lambda(s)$ $s \in S$ considered as a root of $p_\xi(\lambda) = 0$ is a constant in a neighborhood of s if and only if

$$\partial_\lambda p_{\rho(s)}(\lambda(s)) = 0.$$

Let Γ the real part of S. Thus, Γ involves the (real) coordinates ξ_1, ξ_2. If $s \in S$ such that $\xi := \rho(s) = (\xi_1, \xi_2) \in \Gamma$, then

$$\partial_\lambda p_\xi(\lambda(s)) = \langle \xi, \nabla_\xi(q(\lambda\xi) \rangle.$$

The singular points of $\{q(\xi) = 0\} \subset H$ correspond to singular points on S. If we restrict to the set

$$\mathbb{S}^1_\Gamma = \{\xi = (\xi_1, \xi_2) \in \Gamma \ : \ |\xi|^2 = 1\}$$

and suppose that the variety $\{q = 0\}$ is real, then the singular points ξ of the set $\Gamma^{-1}(\mathbb{S}^1_\Gamma)$ have the property such that the line $z\xi$ is tangent to $\{q = 0\}$ or passes through ξ.

The projection map $\rho : S \to \Gamma$ is C^∞ and thus defines the pullback

$$\rho^* : \Omega^2(\Gamma) \to \Omega^2(S).$$

For the differential form

$$\omega := \frac{1}{\partial_\lambda p_\xi(\lambda)} d\xi_1 \wedge d\xi_2,$$

we have that it is well defined on the non singular points of $\rho^{-1}(\Gamma)$. The restriction of ω to S_Γ^1 is

$$\omega' = \frac{1}{\partial_\lambda p_\xi(\lambda)} \rho^*(d\sigma(\xi)).$$

The form $p_\xi(\lambda)$ is exact on S. This shows us that the differential form ω is singular at those points ξ such that

$$\nabla_\xi(q) := \left(\frac{\partial q}{\partial \xi_1}, \frac{\partial q}{\partial \xi_2}\right) = (0,0).$$

Let $q \in \mathbb{C}[\xi_1, \xi_2]$ and let $p \in \mathbb{R}$.

Case 1: q is homogenous$v$ behaves like $e^{P_m(\xi_1,\xi_2)}$ as $P_m \to 0$.

If q is a homogenous (of degree m) polynomial such that $V := \{q = 0\}$ has no singular points, v is given by

$$v(x,\xi,p) = \frac{(\langle x,\xi\rangle - p)^{m-1}}{(m-1)!q(\xi)}.$$

Let $\phi \in \mathcal{S}(\mathbb{R}^2)$.
For $\xi \in \mathbb{S}^1_\Gamma$, $\xi \notin V$, by Green's identity:

$$\int_{\langle \xi,x\rangle = p} \phi(x)d\sigma(x) = \int_{\langle \xi,\xi\rangle \geq p} v(x,\xi,p)q^*\left(\frac{\partial}{\partial x}\right)\phi(x)d\sigma(x).$$

Also, $\phi(z) =$

$$= c_m\Delta_z \int_{\mathbb{S}^1_\Gamma} \frac{d\sigma(\xi)}{q(\xi)} \int_0^\infty \log|p - \langle \xi, z\rangle| dp \int\limits_{\langle \xi,z\rangle - p \geq 0} (\langle \xi, z\rangle - p)^{m-1} q(\partial_x)\phi(x)dx,$$

where

$$c_m = \frac{-2}{2\pi i(m-1)!}.$$

Then,

$$F_z(\phi) = c_m FP \int_{\xi_1^2+\xi_2^2=1} \frac{H(\phi;\xi_1,\xi_2,z)}{q(\xi_1,\xi_2)} d\sigma(\xi_1,\xi_2),$$

where

$$H(\phi;\xi,z)=\Delta_z\int_0^\infty \log|p-\langle\xi,z\rangle|dp\int_{\langle\xi,z\rangle-p\geq 0}(\langle\xi,z\rangle-p)^{m-1}\phi(x)dx.$$

We can take the finite part of the above integral since it is actually 1-dimensional and we define the its finite part in the usual way. Then,

$$F_z(q(\partial_x))=\delta.$$

Our distribution $FP(\frac{1}{q})$ is given by

$$\langle FP(\frac{1}{q}),\phi\rangle=\frac{1}{2\pi^2(m-1)!}FP\int_{\mathbb{S}^1_\Gamma}\frac{H(\widehat{\phi};\xi,z)}{q(\xi)}d\sigma(\xi).$$

This is a tempered distribution and thus its fourier transform exists. The Fourier transform gives us the duality between the finite part and our elementary solution:

$$\mathcal{F}(q(\partial_x))=q(-2\pi i).$$

Case 2: q is non-homogenous but irreducible

Fix $\phi \in \mathcal{S}(\mathbb{R}^2)$. For $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, the function

$$p \mapsto \int_{\langle \xi, x\rangle \leq p} e^{\lambda(\langle \xi, x\rangle - p)} \phi(x) dx$$

is tempered for $\lambda \in \mathbb{C}$ such that $Re\lambda \geq 0$. Set $S_1 = \{\langle \xi, x\rangle \geq p\}$ and $S_2 = \{\langle \xi, x\rangle \leq p\}$. then, the following functions are well defined:

$$H_1(\phi; s, z) = \frac{-2}{(2\pi i)^2} \Delta_z \int_{[0,\infty)} \log |p - \langle \xi, x\rangle| dp \int e^{\lambda(\langle \xi, x\rangle - p)} \phi(x) \chi_{S_1}(x) dx$$

and

$$H_2(\phi; s, z) = \frac{-2}{(2\pi i)^2} \Delta_z \int_{[0,\infty)} \log |p - \langle \xi, x\rangle| dp \int e^{\lambda(\langle \xi, x\rangle - p)} \phi(x) \chi_{S_2}(x) dx.$$

We then define

$$H(\phi; s, z) = \begin{cases} H_1(\phi; s, z) & \textit{for } Re\lambda(s) \leq 0 \\ H_2(\phi; s, z) & \textit{for } Re\lambda(s) > 0 \end{cases}$$

Here, $\mathbb{S}^1_\Gamma := \{(\xi_1, \xi_2) \in \Gamma \ : \ \xi_1^2 + \xi_2^2 = 1\} = C(0, 1)$. If the set $\rho^{-1}(\mathbb{S}^1_\Gamma)$ is not singular, then our elementary solution is given by

$$u_z(\phi) = \int_{\rho^{-1}(\mathbb{S}^1_\Gamma)} \frac{H(\phi; s, z)}{\partial_\lambda p_\xi(\lambda)} \rho^*(d\sigma(\xi)), \tag{5}$$

since then the integral is convergent and need not take its finite part. In the case where we have singular points, there is trouble defining the finite part of the above integral. If, however, we assume that it can be defined, (5) gives an elementary solution which is in fact tempered.

The surface S is of real dimension 4 and the set

$$\{\lambda : \partial_\lambda p_\xi(\lambda) = 0\}$$

is a submanifold of S of real dimension ≤ 2. The set $\rho^{-1}(\mathbb{S}^1_\Gamma) \subset S$ has real dimension 1, thus its singular points are isolated. Around the singular points, we can define the finite part of the above integral. Let $s_0 = (\xi_1^0, \xi_2^0, \lambda) \in S$ singular, that is

$$\partial_\lambda p_\xi(\lambda) = \partial_\lambda q(\lambda\xi_1^0, \lambda\xi_2^0) = 0.$$

Without loss of generality, we may assume that $\xi_1^0 \neq 1$ and therefore parameterize the set $\mathbb{S}^1_\Gamma$: the surface is given locally by the set of points $(\xi_1, g(\xi_1)) = (\xi_1, \sqrt{1-\xi_1^2})$. The (one real variable) function $g(\xi_1) = \sqrt{1-\xi_1^2}$ can be represented as a convergent power series in a neighborhood of ξ_1^0. i.e. there exists a $\epsilon > 0$ such that

$$g(\xi_1) = \sum_{i=0}^{\infty} c_i(\xi_1 - \xi_1^0)^i \quad |\xi_1 - \xi_1^0| < \epsilon.$$

Then, the singular point lies on the set $\{\lambda : q(\lambda\xi_1, \lambda\sqrt{1-\xi_1^2})\}$. From Theorem, the complex variable λ can be represented as a convergent power series along each branch A of $\rho^{-1}(\mathbb{S}^1_\Gamma)$:

$$\lambda(\xi_1, \xi_2) = \sum_{k=0}^{\infty} a_k(\xi_1 - \xi_1^0)^{\frac{k}{m}}, \quad |\xi_1 - \xi_1^0| < \epsilon,$$

where $m \in \mathbb{Z}_+$. We then have that

$$\begin{aligned} \frac{1}{\partial_\lambda p_\xi(\lambda)} &= \sum_{i=-\ell}^{\infty} c_i(\xi_1 - \xi_1^0)^{\frac{i}{\mu m}} \\ &= c_{-\ell}(\xi_1 - \xi_1^0)^{\frac{-\ell}{\mu m}} + \ldots + c_{-1}(\xi_1 - \xi_1^0)^{\frac{-1}{\mu m}} + c_0 + \\ &\quad + c_1(\xi_1 - \xi_1^0)^{\frac{1}{\mu m}} + \ldots \ . \end{aligned}$$

Case 3: q is a random polynomial

Let $q \in \mathbb{R}[x, y]$. Write

$$q = \prod_{i=1}^{\nu} q_i^{k_i},$$

where the q_i's are irreducible. Let $V = \{q = 0\}$. Without l.o.g. assume that $q(0,0) \neq 0$. Set

$$p_{(x,y)}(\lambda) := q(\lambda x, \lambda y) \quad and \quad p^i_{(x,y)}(\lambda) := q_i(\lambda x, \lambda y) \quad (i = 1, 2, \ldots, \nu).$$

Let H be the surface defined by

$$H(x, y) = \prod_{\substack{i,j=1,\ldots,\nu \\ i<j}} Res(p^i(\lambda), p^j(\lambda))\Delta(p^j(\lambda)) \quad ((x, y) \in \mathbb{R}^2). \qquad (6)$$

Denote with $Sing(q)$ the set of all points (x, y) on the plane that q cannot be written as a power of an analytic function. At the points away from $Sing(q)$, we can define the Finite part of $\frac{1}{q}$. This is the main result of the following Theorem:

Theorem

Let $q \in \mathbb{R}[x, y]$ and $V := \{q = 0\}$. Let $\epsilon > 0$. Let $\phi \in \mathcal{S}(\mathbb{R}^2)$ fixed. Then, the expression

$$\int_{S(\epsilon)} \frac{\phi(x, y)dxdy}{q(x, y)},$$

where $S(\epsilon) = \{|q| > \epsilon\} \cap (\mathbb{R}^2 \setminus Sing(q))$, is of the form

$$\sum_{i \in I} \frac{c_i}{\epsilon^i} + c \log(\epsilon) + F(\epsilon), \tag{7}$$

were I is a finite set of natural numbers, c_i and c are complex numbers and F a function such that the $\lim_{\epsilon \to 0} F(\epsilon)$ exists.

Lemma

Let $(x, y) \in (\mathbb{R}^2 \setminus V) \cap \{H \neq 0\}$. *Let* λ_i, $1 \leq i \leq M$ *be the distinct roots of the equation* $p_\lambda(x, y) = 0$ *with corresponding multiplicities* m_k. *Then the function*

$$\lambda \mapsto \frac{1}{p_{(x,y)}(\lambda)(\lambda - 1)}$$

has poles of order m_k *at the points* λ_k. *Moreover we have*

$$\frac{1}{q(x, y)} = -\sum_{k=1}^{M} Res\left(\frac{1}{p_{(x,y)}(\lambda)(\lambda - 1)}; \lambda = \lambda_k\right).$$

We just mention that each of the above Residues is of the form

$$\frac{c_k x^\ell y^\mu \lambda_k^\nu}{[\frac{\partial^{m_k}}{\partial \lambda^{m_k}} p_{(x,y)}(\lambda_k)(\lambda_k - 1)]^{m_k}},$$

where c_k is a constant and ℓ, μ, ν are positive integers. Since we are interested in the preimage of H with the function ρ restricted on $\mathbb{S}^1_\Gamma = C(0, 1)$, we must consider $\rho^*(H)$.

For this reason, we parameterize the surface $\{H = 0\}$ with polar coordinates: let $g : [0, \infty) \times [0, 2\pi] \to \mathbb{R}^2$ with $g(t, \theta) = (t \cos\theta, t \sin\theta)$. Then,

$$(g^{-1})^*(H) = \{\theta - \theta_i = 0 \ : \ 1 \leq i \leq \ell\},$$

for some $\ell \in \mathbb{N}$. Let $\phi \in \mathcal{S}(\mathbb{R}^2))$ fixed. Let $G(\delta) = \{(t\cos\theta, t\sin\theta) : |\theta - \theta_i| > \delta, \ 1 \leq i \leq \ell\}$. Denote with $G(\delta, \epsilon) = G(\delta) \cap \{|q| > \epsilon\}$. Then, define

$$g_\delta(\phi; \epsilon) := \int_{G(\delta,\epsilon)} \frac{\phi(x, y)}{q(x, y)} d\sigma(x, y).$$

This function satisfies the conditions of Theorem (**??**) and thus we can perform integration by parts and write

$$g_\delta(\phi; \epsilon) = \sum_{i \in I} \frac{c_i}{\epsilon^i} + c \log(\epsilon) + F_1(\epsilon), \tag{8}$$

were I is a finite set of natural numbers, c_i and c are complex numbers and $F_1(\epsilon)$ a function such that the $\lim_{\epsilon \to 0} F_1(\epsilon)$ exists.

Set

$$u_\delta(\phi) = FP \lim_{\delta \to 0} F_1(\epsilon, \delta).$$

In what follows, we show that (for $\delta > 0$ and ϕ fixed) we can integrate by parts the function $u_\delta(\phi)$ and obtain

$$u_\delta(\phi) = \sum_{i \in I} \frac{c_i}{\epsilon^i} + c \log(\epsilon) + F_2(\delta), \tag{9}$$

were I is a finite set of natural numbers, c_i and c are complex numbers and $F_2(\delta)$ a function such that the $\lim_{\delta \to 0} F_2(\delta)$ exists. Then, we prove that

$$u(\phi) = FP \lim_{\delta \to 0} F_2(\delta)$$

is a tempered distribution which is in fact a fundamental solution to our division problem.

We have:

$$g_\delta(\phi;\epsilon) = \int_{G(\delta,\epsilon)} \frac{\phi(x,y)}{q(x,y)} d\sigma(x,y)$$
$$= -\sum_{k=1}^{M} \frac{c_k x^\ell y^\mu \lambda_k^\nu}{[(\lambda_k - 1)\frac{\partial^{m_k}}{\partial \lambda^{m_k}} p_{(x,y)}(\lambda_k)]^{m_k}}.$$

Under the transformation $(g^{-1})^*$ restricted on $C(0,1)$, we have

$$(g^{-1})^* \frac{c_k x^\ell y^\mu \lambda_k^\nu}{[(\lambda_k - 1)\frac{\partial^{m_k}}{\partial \lambda^{m_k}} p_{(x,y)}(\lambda_k)]^{m_k}} = \frac{c_k x^\ell(\theta) y^\mu(\theta) \lambda_k^\nu(\theta)}{[(\lambda_k(\theta) - 1)\frac{\partial^{m_k}}{\partial \lambda^{m_k}} p_{(x(\theta),y(\theta))}(\lambda_k(\theta))]^{m_k}}$$

for $(x,y) \in C(0,1)$. Therefore, we finally get

$$g_\delta(\phi;\epsilon) = -\sum_{k=1}^{M} \int_{C(0,1)\cap G(\delta)} R(\theta) d\theta \int_{[0,\infty)\cap\{|(q\circ g)(t,\theta)|>\epsilon\}} \frac{t^\alpha (\phi\circ g)(t,\theta)}{(\lambda_k(\theta) - t)^{m_k}} dt, \tag{10}$$

where $\alpha = \ell + \mu - \nu + m_k(1 - m_k) + 1$ and

$$R(\theta) = \frac{c_k x^{\ell}(\theta) y^{\mu}(\theta) \lambda_k^{\nu}(\theta)}{[\frac{\partial^{m_k}}{\partial \lambda^{m_k}} p_{(x(\theta),y(\theta))}(\lambda_k(\theta))]^{m_k}}.$$

We need only to study one of the integrals

$$\int_{C(0,1)\cap G(\delta)} R(\theta) d\theta \int_{[0,\infty)\cap\{|(q\circ g)(t,\theta)|>\epsilon\}} \frac{t^{\alpha}(\phi \circ g)(t,\theta)}{(\lambda_k(\theta) - t)^{m_k}} dt \equiv J_k(\epsilon;\phi).$$

Now, as we saw in a previous section, since the hypersurface $\{H = 0\}$ contains all singular points of q, then we can interchange summation with finite parts (this is done buy splitting the surface into components and integrating by parts):

$$\int_{[0,\infty)\cap\{|(q\circ g)(t,\theta)|>\epsilon\}} \frac{t^{\alpha}(\phi \circ g)(t,\theta)}{(\lambda_k(\theta) - t)^{m_k}} dt = \sum_{i\in I} \frac{c_i(\theta)}{\epsilon^i} + c(\theta)\log\epsilon + w(\epsilon,\theta),$$

where w is continuous for both $\epsilon,\ \theta$.

Then,

$$J_k(\epsilon;\phi) = \frac{1}{\epsilon^i}\sum_{i\in I}\int_{C(0,1)\cap G(\delta)} R(\theta)c_i(\theta)d\theta + \\ + \log\epsilon\int_{C(0,1)\cap G(\delta)} R(\theta)c(\theta)d\theta + \int_{C(0,1)\cap G(\delta)} R(\theta)w(\epsilon,\theta)d\theta.$$

Since w is continuous, we have

$$\lim_{\epsilon\to 0}\int_{C(0,1)\cap G(\delta)} R(\theta)w(\epsilon,\theta)d\theta = \int_{C(0,1)\cap G(\delta)} R(\theta)\lim_{\epsilon\to 0} w(\epsilon,\theta)d\theta$$

and thus we can take

$$FP\lim_{\epsilon\to 0}\int_{C(0,1)\cap G(\delta)} R(\theta)w(\epsilon,\theta)d\theta = FP\int_{C(0,1)\cap G(\delta)} R(\theta)\lim_{\epsilon\to 0} w(\epsilon,\theta)d\theta.$$

What is left is to integrate by parts the integral appearing on the right side of the above formula and repeat the same process to obtain a limit as $\delta \to 0$ for the FP in order to construct the distribution. This is done by expanding the function as a Taylor series around $t = 0$ and by expanding $\lambda_k(\theta)$ as a power series in indeterminates of x or y, along the unit circle. The continuity of the distributional operator is guaranteed by proving that one can interchange differentiation (with respect to x, y, λ) and integration.

Important Lemma Let $q \in \mathbb{R}[x, y]$. Let $\xi = (\xi_1, \xi_2)$ such that $\xi_1^2 + \xi_2^2 = 1$. Consider the set

$$\{\lambda \in \mathbb{C} \ : \ q(\lambda\xi_1, \lambda\xi_2) = 0\}.$$

Let $\lambda_k(\xi)$ be a branch belonging on this set.

Lemma

If $\xi^0 \in C(0,1)$ with $|\xi|^2 = 1$, let ξ_i, $(j = 1, 2)$ be a coordinate function for $C(0,1)$ around ξ^0. Then, there exists $h \in \mathbb{Z}_+$, a determination z of $(\xi_j - \xi_j^0)^{\frac{1}{h}}$ and a $\tau > 0$ such that

$$\lambda_k(\xi) = \sum_{i=0}^{\infty} c_i z^i, \quad (|z| < \tau)$$

where the above series is convergent.

$n = 3$

Let $q \in \mathbb{R}[x, y, x]$ of degree m with complex coefficients. Write

$$q(\xi) = q_0(\xi) + q_1(\xi) + \ldots + q_m(\xi),$$

where each q_i is a homogenous polynomial of degree i, $(0 \leq i \leq m)$. Let $p \in \mathbb{C}$ and $x \in \mathbb{C}^3$.

Assume that $q_m(\xi) \neq 0$ forall ξ. **The irreducible case**

In the special case where q is homogenous (of degree m) and the variety $V = \{q = 0\}$ has no singular points except at the origin, we can easily calculate (as we did in the previous section) v :

$$v(x, \xi, p) = \frac{(\langle x, \xi \rangle - p)^{m-1}}{(m-1)! q(\xi)}. \tag{11}$$

Let $\phi \in \mathcal{S}(\mathbb{R}^2)$ fixed. We have that

$$\phi(z) = \frac{\Delta_z}{2(2\pi i)^2 (m-1)!} \int_{\mathbb{S}^2_\Gamma} \frac{d\sigma(\xi)}{q(\xi)} \int_{\{\langle \xi, x-z \rangle \geq 0\}} (\langle \xi, x-z \rangle)^{m-1} q^*(\partial_x) \phi(x) dx.$$

Then, the formal expression for an elementary solution is given by

q non-homogenous Remember that a point ξ is a singular point of the variety $\{q(\xi)=0\}$ if

$$\frac{\partial q}{\partial \xi_1}(\xi)=\frac{\partial q}{\partial \xi_2}(\xi)=\frac{\partial q}{\partial \xi_3}(\xi)=0.$$

In the non-homogenous case, the elementary solution is given by

$$u_z(\phi)=FP\int_{\rho^{-1}(\mathbb{S}^2_\Gamma)}\frac{H(\phi;t,z)}{\partial_\lambda p_\xi(\lambda)}\rho^*(d\sigma(\xi)), \tag{13}$$

where

$$H(\phi;t,z)=\begin{cases} H_1(\phi;t,z) & for\ Re\lambda(s)\leq 0\\ H_2(\phi;t,z) & for\ Re\lambda(s)>0\end{cases}$$

$$H_1(\phi;t,z)=\frac{-2}{(2\pi i)^2}\Delta_z\int e^{\lambda(t)(\langle\xi,x\rangle-z)}\phi(x)\chi_{S_1}(x)dx$$

and

$$H_2(\phi;t,z)=\frac{-2}{(2\pi i)^2}\Delta_z\int e^{\lambda(\langle\xi,x\rangle-p)}\phi(x)\chi_{S_2}(x)dx$$

for $S_1=\{\langle\xi,x-t\rangle\geq 0\}$ and $S_2=\{\langle\xi,x-t\rangle<0\}$.

If the Finite Parts of the above integrals are defined, then this is a tempered elementary solution. If we do not require that this is a tempered solution, then we can give conditions under which the elementary solution is well defined. Such conditions are for example that each point of $\mathbb{S}^2_\Gamma \cap \{q_m = 0\}$ has $m-$distinct roots, one of each is necessarily infinite.

Theorem

Let $q \in \mathbb{R}[x, y, z]$ with real coefficients and let $V := \{q = 0\}$. Let

$$D = \{(x, y, z) \in \mathbb{R}^3 \ : \ |\nabla q(x, y, z)| > 0\} \cap V$$

Let $\alpha \in \mathbb{N}$. Let $\epsilon > 0$. Then then correspondence

$$\mathcal{S}(\mathbb{R}^3) \to \mathbb{R} \quad with \quad \phi \mapsto \int_{\{|q|>\epsilon\}\cap D} \frac{\phi(x, y, x)}{q^\alpha(x, y, z)} dxdydz$$

is of the form

$$\sum_{i \in I} \frac{c_i}{\epsilon^i} + c \log(\epsilon) + F(\epsilon), \tag{14}$$

were I is a finite set of natural numbers, c_i and c are complex numbers

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