Δραστηριότητες σελ. 51 (Ενότητα 3.6.6: Ολοκλήρωση ρητών συναρτήσεων)

1.

(α)
$$\int \frac{6x}{x^2-4}dx = 6\int \frac{xdx}{(x-2)(x+2)}dx = \frac{6}{2}\int \left(\frac{1}{x-2}+\frac{1}{x+2}\right)dx = 3(\ln|x-2|+\ln|x+2|)+c$$
$$= 3\ln|x^2-4|+c$$

(β)
$$\int \frac{2x-7}{x^2-7x+3}dx = \int \frac{(x^2-7x+3)'}{x^2-7x+3}dx = \ln|x^2-7x+3|+c$$

(γ)
$$\int \frac{1}{x^2+2x}dx = \int \frac{1}{x(x+2)}dx = \frac{1}{2}\int \left(\frac{1}{x}-\frac{1}{x+2}\right)dx = \frac{1}{2}\ln\left|\frac{x}{x+2}\right|+c$$

(δ)
$$\int \frac{2}{x^2-9}dx = 2\int \frac{dx}{(x-3)(x+3)}dx = \frac{2}{6}\int \left(\frac{1}{x-3}-\frac{1}{x+3}\right)dx = \frac{1}{3}\ln\left|\frac{x-3}{x+3}\right|+c$$

(ε)
$$\int \frac{5x+4}{x^2+x-2}dx = \int \frac{5x+4}{(x-1)(x+2)}dx = \int \left(\frac{3}{x-1}+\frac{2}{x+2}\right)dx = 3\ln|x-1|+2\ln|x+2|+c$$

(στ)
$$\int \frac{6x-4}{x^2-6x+13}dx = \int \frac{6x-18+14}{x^2-6x+13}dx = \int \frac{6x-18}{x^2-6x+8}dx + 14\int \frac{dx}{x^2-6x+13}$$

Είναι
$$\int \frac{6x-18}{x^2-6x+13}dx = 3\int \frac{2x-6}{x^2-6x+13}dx = 3\int \frac{d(x^2-6x+13)}{x^2-6x+13}dx = 3\ln(x^2-6x+13)+c_1$$

και
$$\int \frac{dx}{x^2-6x+13} = \int \frac{dx}{(x-3)^2+4} = \frac{1}{4}\int \frac{dx}{\left(\frac{x-3}{2}\right)^2+1} = \frac{1}{2}\int \frac{d\left(\frac{x-3}{2}\right)}{\left(\frac{x-3}{2}\right)^2+1} = \frac{1}{2}\tau o\xi \varepsilon\varphi\left(\frac{x-3}{2}\right)+c_2$$

Συνεπώς, $\boldsymbol{\int \frac{6x-4}{x^2-6x+13}dx = 3\ln(x^2-4x+13)+7\tau o\xi\varepsilon\varphi\left(\frac{x-3}{2}\right)+c}$

(ζ)
$$\int \frac{x^3+2x+6}{x^2+x-2}dx = \int \left(x-1+\frac{5x+4}{x^2+x-2}\right)dx = \int (x-1)\,dx + \int \frac{5x+4}{x^2+x-2}dx$$
$$= \frac{x^2}{2}-x+\int \frac{5x+4}{(x-1)(x+2)}dx = \frac{x^2}{2}-x+\int \left(\frac{3}{x-1}+\frac{2}{x+2}\right)dx$$
$$\boldsymbol{= \frac{x^2}{2}-x+3\ln|x-1|+2\ln|x+2|+c}$$

(η)
$$\int \frac{5x}{(x^2+4)(x+1)}dx = \int \left(\frac{x+4}{x^2+4}-\frac{1}{x+1}\right)dx = \int \frac{x}{x^2+4}dx + 4\int \frac{dx}{x^2+4} - \int \frac{1}{x+1}dx$$
$$= \frac{1}{2}\int \frac{(x^2+4)'}{x^2+4}dx + 2\int \frac{d\left(\frac{x}{4}\right)}{\left(\frac{x}{4}\right)^2+1} - \ln|x+1| = \boldsymbol{\frac{1}{2}\ln(x^2+4)+2\tau o\xi\varepsilon\varphi\left(\frac{x}{2}\right)-\ln|x+1|+c}$$

(θ)
$$\int \frac{2x}{x^2-2x+10}dx = \int \frac{2x-2+2}{x^2-2x+10}dx = \int \frac{2x-2}{x^2-2x+10}dx + 2\int \frac{dx}{x^2-2x+10}$$

Είναι
$$\int \frac{2x-2}{x^2-2x+10}dx = \int \frac{d(x^2-2x+10)}{x^2-2x+10}dx = \ln(x^2-2x+10)+c_1$$

και
$$\int \frac{dx}{x^2-2x+10} = \int \frac{dx}{(x-2)^2+9} = \frac{1}{9}\int \frac{dx}{\left(\frac{x-1}{3}\right)^2+1} = \frac{1}{3}\int \frac{d\left(\frac{x-1}{3}\right)}{\left(\frac{x-1}{9}\right)^2+1} = \frac{1}{3}\tau o\xi\varepsilon\varphi\left(\frac{x-1}{3}\right)+c_2$$

Συνεπώς, $\boldsymbol{\int \frac{2x}{x^2-2x+10}dx = \ln(x^2-2x+10)+\frac{2}{3}\tau o\xi\varepsilon\varphi\left(\frac{x-1}{3}\right)+c}$

(ι) $$\int \frac{dx}{x^2+10x+29} = \int \frac{dx}{(x+5)^2+4} = \frac{1}{4}\int \frac{dx}{\left(\frac{x+5}{2}\right)^2+1} = \frac{1}{2}\int \frac{d\left(\frac{x+5}{2}\right)}{\left(\frac{x+5}{2}\right)^2+1}$$
$$= \boldsymbol{\frac{1}{2}\tau o\xi\varepsilon\varphi\left(\frac{x+5}{2}\right)} + c$$

(ια) $$\int \frac{dx}{x^3(x+1)} = \int\left(\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}-\frac{1}{x+1}\right)dx = \int \frac{1}{x}dx - \int\frac{1}{x^2}dx + \int\frac{1}{x^3}dx - \int\frac{dx}{x+1}$$
$$= \ln|x| + \frac{1}{x} - \frac{1}{2x^2} - \ln|x+1| + c = \boldsymbol{\frac{1}{x} - \frac{1}{2x^2} + \ln\left|\frac{x}{x+1}\right| + c}$$

(ιβ) $$\int \frac{dx}{x^4-1} = \int \frac{dx}{(x^2-1)(x^2+1)} = \int \frac{dx}{(x-1)(x+1)(x^2+1)}$$
$$= \frac{1}{2}\int\left(\frac{1}{2(x-1)} - \frac{1}{2(x+1)} - \frac{1}{x^2+1}\right)dx = \boldsymbol{\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\tau o\xi\varepsilon\varphi(x) + c}$$

(ιγ) $$\int \frac{1}{x(x^2+1)^2}dx = \int\left(\frac{1}{x} - \frac{x}{x^2+1} - \frac{x}{(x^2+1)^2}\right)dx = \boldsymbol{\ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + c}$$

(ιδ) Θεωρούμε την αντικατάσταση $u = \varepsilon\varphi\left(\frac{x}{2}\right)$ και το πιο πάνω μετασχηματίζεται στο
$$\int \frac{10}{3\eta\mu x + 4\sigma\upsilon\nu x}dx = -\int \frac{du}{2u^2-3u-2} = -\int\frac{du}{(u-2)(2u+1)} = -\frac{1}{5}\int\left(\frac{1}{u-2} - \frac{2}{2u+1}\right)du$$
$$= \frac{1}{5}\ln\left|\frac{2u+1}{u-2}\right| + c = \boldsymbol{\frac{1}{5}\ln\left|\frac{2\varepsilon\varphi\left(\frac{x}{2}\right)+1}{\varepsilon\varphi\left(\frac{x}{2}\right)-2}\right| + c}$$

(ιε) $$\int \frac{8}{3+5\eta\mu(2x)}dx = 8\int\frac{dx}{3+5\eta\mu(2x)}$$

Θεωρούμε την αντικατάσταση $u = \varepsilon\varphi x \Rightarrow x(u) = \tau o\xi\varepsilon\varphi u \Rightarrow dx = \frac{du}{1+u^2}$. Επίσης,
$$\eta\mu(2x) = \frac{2\varepsilon\varphi x}{1+\varepsilon\varphi^2 x} = \frac{2u}{1+u^2}$$
και αρα
$$\int\frac{8dx}{3+5\eta\mu(2x)} = 8\int\frac{\frac{du}{1+u^2}}{3+5\frac{2u}{1+u^2}} = 8\int\frac{du}{3u^2+10u+3} = 8\int\frac{1}{8}\left(\frac{3}{3u+1} - \frac{1}{u+3}\right)dt$$
$$= \int\left(\frac{3}{3u+1} - \frac{1}{u+3}\right)du = (\ln|3u+1| - \ln|u+3|) + c = \ln\left|\frac{3u+1}{u+3}\right| + c = \boldsymbol{\ln\left|\frac{3\varepsilon\varphi x+1}{\varepsilon\varphi x+3}\right| + c}$$

(ιστ) $\int\frac{1}{5+3\sigma\upsilon\nu x}dx$. Θεωρούμε την αντικατάσταση $u = \frac{1}{2}\varepsilon\varphi\left(\frac{x}{2}\right)$ και προχωρώντας *εντελώς* όμοια με τα 2 προηγούμενα, βρίσκουμε $\int\frac{1}{5+3\sigma\upsilon\nu x}dx = \boldsymbol{\frac{1}{2}\tau o\xi\varepsilon\varphi\left(\frac{1}{2}\varepsilon\varphi\left(\frac{x}{2}\right)\right) + c}$

(ιγ) $$\int \frac{1}{x(x^2+1)^2}dx = \int\left(\frac{1}{x} - \frac{x}{x^2+1} - \frac{x}{(x^2+1)^2}\right)dx = \boldsymbol{\ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + c}$$

(ιδ) Θεωρούμε την αντικατάσταση $u = \varepsilon\varphi\left(\frac{x}{2}\right)$ και το πιο πάνω μετασχηματίζεται στο
$$\int \frac{10}{3\eta\mu x + 4\sigma\upsilon\nu x}dx = -\int \frac{du}{2u^2-3u-2} = -\int\frac{du}{(u-2)(2u+1)} = -\frac{1}{5}\int\left(\frac{1}{u-2} - \frac{2}{2u+1}\right)du$$
$$= \frac{1}{5}\ln\left|\frac{2u+1}{u-2}\right| + c = \frac{1}{5}\ln\left|\frac{2\varepsilon\varphi\left(\frac{x}{2}\right)+1}{\varepsilon\varphi\left(\frac{x}{2}\right)-2}\right| + c$$

(ιε) $\int \frac{8}{3+5\eta\mu(2x)}dx = 8\int \frac{dx}{3+5\eta\mu(2x)}$

Θεωρούμε την αντικατάσταση $u = \varepsilon\varphi x \Rightarrow x(u) = \tau o\xi\varepsilon\varphi u \Rightarrow dx = \frac{du}{1+u^2}$. Επίσης,

$$\eta\mu(2x) = \frac{2\varepsilon\varphi x}{1+\varepsilon\varphi^2 x} = \frac{2u}{1+u^2}$$

και αρα

$$\int \frac{8dx}{3+5\eta\mu(2x)} = 8\int \frac{\frac{du}{1+u^2}}{3+5\frac{2u}{1+u^2}} = 8\int \frac{du}{3u^2+10u+3} = 8\int \frac{1}{8}\;\frac{3}{3u+1} - \frac{1}{u+3}\Big) dt$$

$$= \int \left(\frac{3}{3u+1} - \frac{1}{u+3}\right) du = (\ln|3u+1| - \ln|u+3|) + c = \ln\left|\frac{3u+1}{u+3}\right| + c = \ln\left|\frac{3\varepsilon\varphi x+1}{\varepsilon\varphi x+3}\right| + c$$

(ιστ) $\int \frac{1}{5+3\sigma\upsilon\nu x}dx$. Θεωρούμε την αντικατάσταση $u = \frac{1}{2}\varepsilon\varphi\left(\frac{x}{2}\right)$ και προχωρώντας *εντελώς* όμοια με τα 2 προηγούμενα, βρίσκουμε $\int \frac{1}{5+3\sigma\upsilon\nu x}dx = \frac{1}{2}\tau o\xi\varepsilon\varphi\left(\frac{1}{2}\varepsilon\varphi\left(\frac{x}{2}\right)\right) + c$