Δραστηριότητες σελ. 41 (Ενότητα 3.6.4: Ολοκλήρωση τριγωνομετρικών συναρτήσεων)

1.

(α) $$\int \eta\mu^2(x)\,dx = \int \frac{1-\sigma\upsilon\nu(2x)}{2}dx = \frac{1}{2}\int 1-\sigma\upsilon\nu(2x)\Big)\,dx = \frac{x}{2}-\frac{1}{4}\eta\mu(2x)+c$$
$$=\frac{1}{2}\big(x-\eta\mu(x)\cdot\sigma\upsilon\nu(x)\big)+c$$

(β) $$\int \sigma\upsilon\nu^3(x)\,dx = \int \sigma\upsilon\nu^2(x)\cdot\sigma\upsilon\nu(x)\,dx = \int \sigma\upsilon\nu^2(x)\,d\big(\eta\mu(x)\big)$$
$$=\int\big(1-\eta\mu^2(x)\big)\,d\big(\eta\mu(x)\big) = \int d\big(\eta\mu(x)\big) - \int \eta\mu^2(x)\,d\big(\eta\mu(x)\big) = \eta\mu(x)-\frac{\eta\mu^3(x)}{3}+c$$

(γ) $$\int \sigma\upsilon\nu^4(x)\,dx = \int\left(\frac{1+\sigma\upsilon\nu(2x)}{2}\right)^2 dx = \frac{1}{4}\int\big(1+\sigma\upsilon\nu(2x)\big)^2\,dx$$
$$=\frac{1}{4}\int\big(1+2\sigma\upsilon\nu(2x)+\sigma\upsilon\nu^2(2x)\big)\,dx = \frac{1}{4}\left(\int dx+\int\sigma\upsilon\nu(2x)d(2x)+\frac{1}{2}\int\left(\frac{1+\sigma\upsilon\nu(2x)}{2}\right)dx\right)$$
$$=\frac{1}{4}\quad\int\frac{3}{2}dx+\int\sigma\upsilon\nu(2x)d(2x)+\frac{1}{8}\int\sigma\upsilon\nu(2x)d(2x)\Big)$$
$$=\frac{1}{4}\left(\frac{3x}{2}+\eta\mu(2x)+\frac{1}{2}\left(\frac{1}{4}\eta\mu(4x)\right)\right)=\frac{3x}{8}+\frac{\eta\mu(2x)}{4}+\frac{\eta\mu(4x)}{32}+c$$

Διαφορετικά,
$$\int\sigma\upsilon\nu^4(x)\,dx = \frac{1}{4}\sigma\upsilon\nu^3(x)\cdot\eta\mu(x)+\frac{1}{2}\int\sigma\upsilon\nu^2 x dx = \cdots$$

(δ) $$\int\frac{\eta\mu^3 x}{\sigma\upsilon\nu x}dx = \int \eta\mu^2 x\frac{\eta\mu x}{\sigma\upsilon\nu x}dx = \int\eta\mu^2 x\varepsilon\varphi x\,dx = \int(1-\sigma\upsilon\nu^2 x)\varepsilon\varphi x\,dx$$
$$=\int\varepsilon\varphi x\,dx-\int\sigma\upsilon\nu^2 x\varepsilon\varphi x\,dx = -\ln|\sigma\upsilon\nu(x)|-\int\sigma\upsilon\nu x\eta\mu x\,dx$$
$$=-\ln|\sigma\upsilon\nu(x)|-\frac{1}{2}\int\eta\mu(2x)\,dx = -\ln|\sigma\upsilon\nu(x)|+\frac{1}{2}\sigma\upsilon\nu(2x)+c$$

(ε) $$\int\eta\mu^4(x)\cdot\sigma\upsilon\nu^3(x)dx = \int\eta\mu^4(x)\cdot\sigma\upsilon\nu^2(x)\cdot\sigma\upsilon\nu(x)dx$$
$$=\int\eta\mu^4(x)\cdot\big(1-\eta\mu^2(x)\big)\cdot d\big(\eta\mu(x)\big)=\int\big(\eta\mu^4(x)-\eta\mu^6(x)\big)\cdot d\big(\eta\mu(x)\big)$$
$$=\frac{\eta\mu^5(x)}{5}-\frac{\eta\mu^7(x)}{7}+c$$

(στ) $$\int\big(\eta\mu^2(x)+\sigma\upsilon\nu^4(x)\big)\,\eta\mu(x)dx = \int\big(1-\sigma\upsilon\nu^2(x)+\sigma\upsilon\nu^4(x)\big)\,\eta\mu x dx$$
$$=-\int\big(1-\sigma\upsilon\nu^2(x)+\sigma\upsilon\nu^4(x)\big)\,d\big(\sigma\upsilon\nu(x)\big)=-\sigma\upsilon\nu(x)+\frac{\sigma\upsilon\nu^3(x)}{3}-\frac{\sigma\upsilon\nu^5(x)}{5}+c$$

(ζ) $\int \left(\eta\mu^3(x)\sigma\upsilon\nu(x) - \eta\mu(x)\sigma\upsilon\nu^3(x)\right) dx = \int \eta\mu(x)\sigma\upsilon\nu(x)\left(\eta\mu^2(x) - \sigma\upsilon\nu^2(x)\right) dx$

$= \frac{1}{2}\int \eta\mu(2x)\sigma\upsilon\nu(2x)\, dx = \frac{1}{4}\int \eta\mu(4x)\, dx = -\frac{1}{16}\sigma\upsilon\nu(4x) + c$

(η) $\int \tau\varepsilon\mu^4(\theta)\, d\theta = \int \tau\varepsilon\mu^2(\theta)\cdot\tau\varepsilon\mu^2(\theta) d\theta = \int \tau\varepsilon\mu^2(\theta)\cdot\left(\varepsilon\varphi(\theta)\right)' d\theta$

$= \tau\varepsilon\mu^2(\theta)\cdot\varepsilon\varphi(\theta) - 2\int \tau\varepsilon\mu^2(x)\cdot\varepsilon\varphi^2(x) d\theta = \tau\varepsilon\mu^2(x)\cdot\varepsilon\varphi(\theta) - 2\int \tau\varepsilon\mu^2(\theta)\cdot(\tau\varepsilon\mu^2(\theta) - 1) d\theta$

$= \tau\varepsilon\mu^2(\theta)\cdot\varepsilon\varphi(\theta) - 2\int \tau\varepsilon\mu^4(\theta)\, d\theta + 2\underbrace{\int \tau\varepsilon\mu^2(\theta)\, d\theta}_{\varepsilon\varphi(\theta)}$

$\Rightarrow 3\int \tau\varepsilon\mu^4(\theta)\, d\theta = \tau\varepsilon\mu^2(\theta)\cdot\varepsilon\varphi(\theta) + 2\varepsilon\varphi(\theta) + c_1$. Έτσι,

$$\int \tau\varepsilon\mu^4(\theta)\, d\theta = \frac{1}{3}\tau\varepsilon\mu^2(\theta)\cdot\varepsilon\varphi(\theta) + \frac{2}{3}\varepsilon\varphi(\theta) + c = \frac{1}{3}\left(1 + \varepsilon\varphi^2(\theta)\right)\cdot\varepsilon\varphi(\theta) + \frac{2}{3}\varepsilon\varphi(\theta) + c$$

$$= \varepsilon\varphi(\theta) + \frac{\varepsilon\varphi^3(\theta)}{3} + c$$

(θ) $\int \tau\varepsilon\mu^3(x)\, dx = \int \tau\varepsilon\mu(x)\cdot\tau\varepsilon\mu^2(x) dx = \int \tau\varepsilon\mu(x)\cdot\left(\varepsilon\varphi(x)\right)' dx$

$= \tau\varepsilon\mu(x)\cdot\varepsilon\varphi(x) - \int \tau\varepsilon\mu(x)\cdot\varepsilon\varphi^2(x) dx = \tau\varepsilon\mu(x)\cdot\varepsilon\varphi(x) - \int \tau\varepsilon\mu(x)\cdot(\tau\varepsilon\mu^2(x) - 1) dx$

$= \tau\varepsilon\mu(x)\cdot\varepsilon\varphi(x) - \int \tau\varepsilon\mu^3(x)\, dx + \int \tau\varepsilon\mu(x)\, dx = \frac{1}{2}\tau\varepsilon\mu(x)\cdot\varepsilon\varphi(x) + \frac{1}{2}\int \tau\varepsilon\mu(x)\, dx$

$= \frac{1}{2}\tau\varepsilon\mu(x)\cdot\varepsilon\varphi(x) + \frac{1}{2}\ln|\tau\varepsilon\mu(x) + \varepsilon\varphi(x)| + c$

(ι) $\int \varepsilon\varphi^3(x)\tau\varepsilon\mu(x)\, dx = \int \varepsilon\varphi^2(x)\varepsilon\varphi(x)\tau\varepsilon\mu(x)\, dx = \int (\tau\varepsilon\mu^2(x) - 1)\underbrace{\varepsilon\varphi(x)\tau\varepsilon\mu(x)}_{(\tau\varepsilon\mu x)'} dx$

$= \int (\tau\varepsilon\mu^2(x) - 1)\, d\left(\tau\varepsilon\mu(x)\right) = \frac{\tau\varepsilon\mu^3(x)}{3} - \tau\varepsilon\mu(x) + c$

(ια) $\int \eta\mu(2x)\cdot\sigma\upsilon\nu(x)\, dx = \frac{1}{2}\int [\eta\mu(2x - x) + \eta\mu(2x + x)]\, dx = \frac{1}{2}\int [\eta\mu(x) + \eta\mu(3x)]\, dx$

$$= -\frac{1}{2}\sigma\upsilon\nu(x) - \frac{1}{6}\sigma\upsilon\nu(3x) + c$$

(ιβ) $\int \eta\mu(5x)\cdot\eta\mu(7x) dx = \frac{1}{2}\int [\sigma\upsilon\nu(5x - 7x) - \sigma\upsilon\nu(5x + 7x)]\, dx = \frac{1}{2}\int [\sigma\upsilon\nu(-2x) - \sigma\upsilon\nu(12x)]\, dx$

$= \frac{1}{2}\int [\sigma\upsilon\nu(2x) - \sigma\upsilon\nu(12x)]\, dx = \frac{1}{4}\eta\mu(2x) - \frac{1}{24}\eta\mu(12x) + c$

(ιγ) $\int \sqrt{1+\eta\mu(2x)}\,dx = \int \sqrt{(\eta\mu x+\sigma\upsilon\nu x)^2}\,dx = \int (\eta\mu x+\sigma\upsilon\nu x)\,dx = \eta\mu x - \sigma\upsilon\nu x + c$

Χρησιμοποιήσαμε το ότι $x \in \left(0, \frac{\pi}{2}\right)$ και το ότι

$(\eta\mu x+\sigma\upsilon\nu x)^2 = \eta\mu^2 x + 2\sigma\upsilon\nu x\mu x + \sigma\upsilon\nu^2 x = 1+\eta\mu(2x)$

Διαφορετικά, $\int \sqrt{1+\eta\mu(2x)}\,dx = \frac{1}{2}\int \sqrt{1+\eta\mu x}\,dx$. Αλλά,

$$1+\eta\mu x = 1+\sigma\upsilon\nu\left(\frac{\pi}{2}-x\right) = 2\sigma\upsilon\nu^2\left(\frac{\pi}{4}-\frac{x}{2}\right) = 2\sigma\upsilon\nu^2\left(\frac{x}{2}-\frac{\pi}{4}\right)$$

και αρα $\frac{1}{2}\int \sqrt{1+\eta\mu x}\,dx = \frac{\sqrt{2}}{2}\int \sqrt{\sigma\upsilon\nu^2\left(\frac{x}{2}-\frac{\pi}{4}\right)}\,dx = \frac{\sqrt{2}}{2}\int \sigma\upsilon\nu\left(\frac{x}{2}-\frac{\pi}{4}\right)dx$

$$-= \sqrt{2}\int \sigma\upsilon\nu\left(\frac{x}{2}-\frac{\pi}{4}\right)d\left(\frac{x}{2}-\frac{\pi}{4}\right) = \sqrt{2}\eta\mu\left(\frac{x}{2}-\frac{\pi}{4}\right)+c = \sqrt{2}\eta\mu\left(\frac{x}{2}-\frac{\pi}{4}\right)+c = \eta\mu x - \sigma\upsilon\nu x + c$$

(ιδ) $\int \frac{\ln(\tau\varepsilon\mu x)}{\sigma\upsilon\nu^2 x}dx = \int \ln(\tau\varepsilon\mu x)\,\tau\varepsilon\mu^2 x\,dx = \int \ln(\tau\varepsilon\mu x)\,(\varepsilon\varphi x)'\,dx$

$= \varepsilon\varphi x\ln(\tau\varepsilon\mu x) - \int \varepsilon\varphi x\,(\ln(\tau\varepsilon\mu x))'dx$ (κατά μέρη)

$= \varepsilon\varphi x\ln(\tau\varepsilon\mu x) - \int \varepsilon\varphi x\,(\ln(\tau\varepsilon\mu x))'dx = \varepsilon\varphi x\ln(\tau\varepsilon\mu x) - \int \varepsilon\varphi x\frac{\tau\varepsilon\mu x\varepsilon\varphi x}{\tau\varepsilon\mu x}dx$

$= \varepsilon\varphi x\ln(\tau\varepsilon\mu x) - \int \varepsilon\varphi^2 x\,dx = \varepsilon\varphi x\ln(\tau\varepsilon\mu x) - (\varepsilon\varphi x - x) + c$

$= \varepsilon\varphi x\ln(\tau\varepsilon\mu x) - \varepsilon\varphi x + x + c$ (Χρησιμοποιήσαμε το ότι $x \in \left(0, \frac{\pi}{2}\right)$)