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Holomorphic functions and the **Wirtinger derivative**

Definition 0.1 (Holomorphic function)**.** If f is (complex) differentiable at every point z_0 of $U \subseteq \mathbb{C}$ open, we say that f is **holomorphic on U**. If f is (complex) differentiable on some (open) neighbourhood of z_0, then we say that f is **holomorphic at $\mathbf{z_0}$**.

The word 'holomorphic' was introduced by two of Cauchy's students, Briot and Bouquet and derives from the Greek words *όλος* (holos) meaning 'entire' or 'whole', and *μορφή* (morphē) meaning 'form' or 'shape'. We shall see later on where this terminology stems from.

Remarks

1. It follows immediately that if f is differentiable at z_0, then it is continuous at z_0.
2. Complex differentiability (or holomorphicity) if often referred to as 'analyticity'. This is due to the fact, as we shall see later, that a holomorphic function can be expanded locally as a (convergent) series, hence the term 'analyticity'. The main tool for proving such a powerful result is Cauchy's Integral formula. Nevertheless, there exists a proof that holomorphicity implies C^∞ without the help of Cauchy's Integral formula. It is based on a topological result by G. T. Whyburn (namely, if f is differentiable and non-constant in a region D of the complex plane, then f is an open map). The proof is presented in [9]. Since in what follows we will study holomorphicity under the scope of differentiability, it is useful that we keep this fact in mind.

Definition 0.2 (Entire function)**.** A function that is holomorphic on the entire complex plane is called an **entire function**.

We shall denote $\mathcal{O}(U)$ the set of all holomorphic functions on U :

$$\mathcal{O}(U) = \{f : U \to \mathbb{C} \,:\, f \,\textit{is holomorphic}\}.$$

Examples

1. The function $f(z) = \overline{z}$, $z \in \mathbb{C}$ is nowhere differentiable.
   Lets first proof that $f'(0)$ does not exist. We have that

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{\overline{h}}{h} = \lim_{\Delta x + i\Delta y \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} \tag{1.1}$$

(where we've written $h = \Delta x + i\Delta y$). If we restrict ourselves in the case were $h \to 0$ along the $x-$axis, i.e. $\Delta x \to 0$, then $\Delta y = 0$ and the above limit then becomes

$$\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

and similarly, if we restrict ourselves in the case were $h \to 0$ along the $y-$axis, i.e. $\Delta y \to 0$, then $\Delta x = 0$ and the above limit then becomes

$$\lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1.$$

Thus, since we approached 0 in two different directions and found the corresponding limits different, we conclude that the limit (1.1) does not exist. For $z \neq 0$, we could follow the same steps as above, but it would perhaps be easier to write z in its polar form: $z_0 = |z_0|e^{iarg(z_0)}$. Then,

$$\begin{aligned} \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} &= \lim_{h \to 0} \frac{\overline{z_0 + h} - \overline{z_0}}{h} = \lim_{h \to 0} \frac{\overline{z_0} + \overline{h} - \overline{z_0}}{h} \\ &= \lim_{h \to 0} \frac{\overline{h}}{h} = \lim_{h \to 0} \frac{|z_0|e^{-iarg(z_0)}}{|z_0|e^{iarg(z_0)}} = \lim_{h \to 0} e^{-2iarg(z_0)} \end{aligned} \qquad (1.2)$$

If $arg(z_0) = 0$ or $= \pi$, i.e. $h \to 0$ along the $x-$axis, then $2iarg(z_0) = 1$ and thus (1.2) equals to 1. Similarly, if $arg(z_0) = \frac{\pi}{2}$ or $= \frac{3\pi}{2}$, i.e. $h \to 0$ along the $y-$axis, then $2iarg(z_0) = -1$ and thus (1.2) equals to -1. The result follows.
Note that since there cannot be an open set around $(0, 0)$ such that f is (complex) differentiable, then

f **is not holomorphic at** $(0, 0)$.

2. Let

$$f(z) = \begin{cases} \frac{\overline{z}^2}{z} & z \neq 0 \\ 0 & z = 0 \end{cases} \qquad (1.3)$$

Obviously, f is continuous in $\mathbb{C} \setminus \{0\}$. Since for $z \neq 0$, $|f(z)| = \left|\frac{\overline{z}^2}{z}\right| = \frac{\overline{|z|}^2}{|z|} = |z|$, we have that $f(z) \to 0 = f(0)$ as $z \to 0$, thus f is continuous at the origin also.
f is (complex) differentiable at the origin:

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|^2}{h} = \lim_{h \to 0} \overline{h} = 0.$$

Thus, $f'(0) = 0$. Viewing the above limit in x, y coordinates, we have

$$\lim_{h=(x,y) \to (0,0)} \frac{|h|^2}{h} = \lim_{(x,y) \to (0,0)} (x - iy) = 0.$$

Now, for $z \neq 0$, we have

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{|z+h|^2 - |z|^2}{h} = \lim_{h \to 0} \frac{z\overline{h} + \overline{z}h}{h}$$

which limit does not exist, since the $\lim_{h\to 0}\frac{\bar{h}}{h}$ does not exist. Otherwise, using x, y coordinates:
• Let $z \to 0$ along the $x-$ axis $(y = 0)$:

$$\lim_{(x,0)\to(0,0)} \frac{f(x+0i)-f(0)}{(x+0i)-0} = \lim_{(x,0)\to(0,0)} \frac{\frac{x^3}{x^2}}{x} = 1$$

• Let $z \to 0$ along the diagonal $(y = x)$: the diagonal is parameterized by $x = t$, $y = x = t$ and thus

$$\lim_{(t,t)\to(0,0)} \frac{f(t+ti)-f(0)}{(t+ti)-0} = \lim_{t\to 0} \frac{-t(t+i)}{t(t+i)} = -1$$

Therefore, f is not differentiable at $z \neq 0$ in the complex sense.

1.1. The Cauchy-Riemann equations

We shall now establish the relationship between the (complex) differentiability of a function $f = u+iv$ and the partial derivatives of u and v. Let f the function defined in ((1.3)). Viewing f as a function from $\mathbb{R}^2$ to $\mathbb{R}$, f is written as $f(x, y) = x^2 + y^2$ and since the partial derivatives of f are continuous functions all over $\mathbb{R}^2$, we conclude that f is differentiable (in $\mathbb{R}^2$). But as we saw earlier, f is (complex) differentiable only at the origin. Thus, the differentiability of u and v does not imply the (complex) differentiability of $f = u + iv$. Also consider the function $f(z) = \bar{z}$, $z \in \mathbb{C}$. Write $f(x + iy) = x - iy$. Then the partial derivatives of all orders of $u(x, y) = x$ and $v(x, y) = -y$ exist but, as we already proved, f is nowhere differentiable. The following Theorem states that the opposite is true: differentiability of $f = u + iv$ implies (partial) differentiability of u and v.

Recall

Definition 1.1 (Smooth Functions)**.** Let $U \subseteq \mathbb{R}^2$ open and $f : U \to \mathbb{R}$ a continuous function. We say that f is of class C^k (or f is $k-$ continuously differentiable, or simply f is C^k) if the partial derivatives of f up to order k and including order k exist and are continuous (functions) on U, (where $k \in \mathbb{N} \cup \{0\}$). In this case, we write $f \in C^k(U)$.

The space $C^0(U)$ is just the space of all continuous functions on U, the space $C^1(U)$ is the space of all differentiable functions on U such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are smooth, $C^2(U)$ is the space of all differentiable functions on U such that $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ $\frac{\partial f^2}{\partial x \partial y}$ and $\frac{\partial f^2}{\partial y \partial x}$ are smooth and so on. The space $C^\infty(U)$ is the space of all functions infinitely differentiable on U.

At this point, lets recall some basic facts about multivariate calculus, restricting ourselves mainly in the bivariate case.

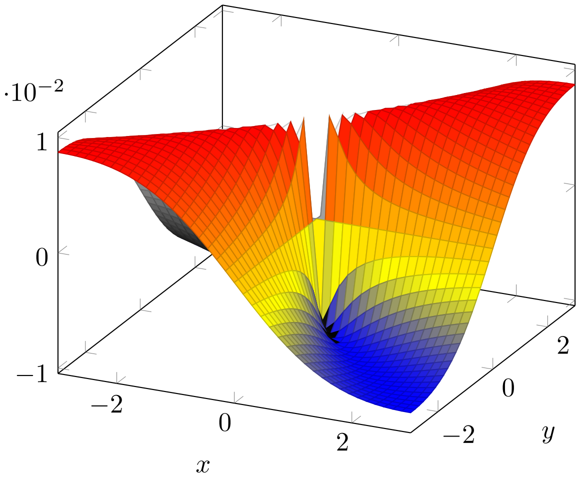
1. **Separately continuous $\not\Rightarrow$ continuity**. It is possible that a function f of two real variables x, y is separately continuous in x, y, at some point (x_0, y_0) of its domain, but not continuous at the point. Consider for example the function

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

We can easily verify that f is continuous as well as separately continuous on $\mathbb{R}^2 \setminus \{(0, 0)\}$. Since $\lim_{x\to 0} f(x, y) = \lim_{y\to 0} f(x, y) = 0 = f(x, y)$, we conclude that f is separately continuous at the point $(0,0)$ also. But f is not continuous at $(0,0)$, for it where, the $\lim_{(x,y)\to(0,0)} f(x,y)$ would equal to $f(0,0)=0$ on every direction. This is not the case on the diagonal $\{y=x\}$:

$$\lim_{(t,t)\to(0,0)} f(x,y) = \lim_{t\to 0}\frac{t^2}{t^2+t^2} = \frac{1}{2}.$$

Here is a plot for the surface $f(x,y)=0$:



2. **Existence of partial derivatives $\not\Rightarrow$ differentiability**.
   It is possible that both partial derivatives of a function f of two real variables x, y exist at some point of its domain, but it is not differentiable there. See the previous example.

3. **Existence of partial derivatives**, **function is continuous $\not\Rightarrow$ differentiability**.
   Example of a function of two real variables x, y where the partial derivatives exist at some point of its domain, is also continuous at that point, but it is not differentiable there:

$$f(x,y) = (xy)^{\frac{1}{3}}.$$

   We immediately see that for $(x,y)\neq(0,0)$

$$\frac{\partial f(x,y)}{\partial x} = \frac{y^{\frac{1}{3}}}{3x^{\frac{2}{3}}} \quad \textit{and (by symmetry)} \quad \frac{\partial f(x,y)}{\partial y} = \frac{x^{\frac{1}{3}}}{3y^{\frac{2}{3}}}$$

   Since $f(x,0)=0$, for $x\neq 0$, $\frac{\partial f(x,0)}{\partial x}=0$, (for $x\neq 0$) but $\frac{\partial f(x,0)}{\partial y}$ does not exist (verify this). By symmetry, the similar results hold if we interchange x and y. Now, since the restriction of f along the $x-$ axis (or the y axis) is identically zero, the partial derivatives $\frac{\partial f(0,0)}{\partial x}$ and $\frac{\partial f(0,0)}{\partial y}$ exist and are both equal to 0. But f is not differentiable at the origin:

Theorem 1.1 (Clairaut-Schwarz)**.** Let $f\in C^2(\mathbb{R}^n)$. Then, for every $(a_1,\ldots,a_n)\in\mathbb{R}^n$

$$\frac{\partial^2 f}{\partial x_i\,\partial x_j}(a_1,\ldots,a_n) = \frac{\partial^2 f}{\partial x_j\,\partial x_i}(a_1,\ldots,a_n) \quad (\forall 1\le i,j\le n).$$

In other words, the partial derivatives of f of order 2 are symmetric.

Theorem 1.2.

Let $D \subseteq \mathbb{R}^2$ a domain. Let $u : D \to \mathbb{R}^2$ such that

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$$

in D. Then u is constant (in D).

Proof. Since in $\mathbb{R}^2$ 'simply connected' is 'equivalent' to 'piecewise connected',we can always consider polygon lines: let $x_0 \in D$ fixed. Let $x \in D$. We can then find a polygon line $\gamma = \gamma_1 \oplus \gamma_2 \ldots \oplus \gamma_N$, (for $N \in \mathbb{N}$) such that every (line segment of this line) γ_i has a parametrization:

$$\gamma_i : [0,1] \to D \quad \text{with} \quad \gamma_i(s) = s \cdot t_i + (1-s)t_{i-1}$$

for $i = 1, 2, \ldots, N$ and $t_0 = x_0$, $t_N = x$. For every i let ϕ_i be the (auxiliary) function defined by

$$\phi_i(s) = u(\gamma_i(s)) = u(s \cdot t_i + (1-s)t_{i-1}), \quad s \in [t_{i-1}, t_i].$$

Then the chain rule gives

$$\phi_i'(s) = \left(\frac{\partial u}{\partial x}(\gamma_i(s)), \frac{\partial u}{\partial y}(\gamma_i(s)) \right) \cdot \gamma_i'(s) = 0$$

and thus, by the Mean Value Theorem of (one variable Calculus) implies that ϕ_i is a constant. Thus $\phi_i(t_i) = \phi_i(t_{i-1})$. Then

$$u(x) - u(x_0) = \sum_{i=1}^{N} (\phi_i(t_i) - \phi_i(t_{i-1})) = 0$$

and therefore u is constant.
Another way to prove this is to define $S = \{u(x) = u(x_0)\}$ and show that it is both a closed and open subset of the (simply connected) set D and thus must be equal to D. □

Theorem 1.3 (Cauchy-Riemann equations)**.** Let $f = u + iv$ defined in a neighborhood of $z_0 = x_0 + iy_0$ and differentiable at z_0. Then, the derivatives

$$\frac{\partial u}{\partial x}(x_0, y_0), \quad \frac{\partial u}{\partial y}(x_0, y_0), \quad \frac{\partial v}{\partial x}(x_0, y_0), \quad \frac{\partial v}{\partial y}(x_0, y_0)$$

exist and

$$f'(x_0 + iy_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0) \tag{1.4}$$

By equating real and imaginary parts in (1.4), we get the following system of equations

$$\left\{ \begin{array}{c} \frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \\ \\ \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0) \end{array} \right\} \tag{1.5}$$

which are called the Cauchy-Riemann equations (C-R for short) for f.

Examples We shall view some of the previous examples under the scope of the previous Theorem.

1. Let $f(z) = |z|^2$, $z \in \mathbb{C}$. Write $f(x+iy) = x^2 + y^2$. Therefore, in this case, $u(x,y) = x^2 + y^2$ and $u(x,y) \equiv 0$. Since u and v are smooth functions (in $\mathbb{R}^2$), it suffices to check the validity of the C-R equations:

$$\left\{ \begin{array}{ll} \dfrac{\partial u}{\partial x} = 2x, & \dfrac{\partial v}{\partial y} = 0 \\ \\ \dfrac{\partial u}{\partial y} = 2y, & \dfrac{\partial v}{\partial x} = 0 \end{array} \right\}$$

The above equations are satisfied only at the origin, hence there can exist no open subset of $\mathbb{C}$ such that f is (complex) differentiable (:f is nowhere analytic).

2. Let $f(z) = \overline{z}$, $z \in \mathbb{C}$. Write $f(x+iy) = x - iy$. $u(x,y) = x$ and $u(x,y) = -y$. As in the previous example, we check the validity of the C-R equations:

$$\left\{ \begin{array}{ll} \dfrac{\partial u}{\partial x} = 1, & \dfrac{\partial v}{\partial y} = 0 \\ \\ \dfrac{\partial u}{\partial y} = -1, & \dfrac{\partial v}{\partial x} = 0 \end{array} \right\}$$

Since the above equations are nowhere satisfied, we conclude that f is nowhere (complex) differentiable.

3. Let

$$f(z) = \begin{cases} \frac{\overline{z}^2}{z} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

For $z = x + iy \neq 0$, we have

$$f(x+iy) = \underbrace{\frac{x^3 - 3xy^2}{x^2+y^2}}_{u(x,y)} + \underbrace{\frac{y^3 - 3x^2y}{x^2+y^2}}_{v(x,y)} i.$$

The C-R equations for f are satisfied at $(0,0)$. To verify this claim, we shall calculate the partial derivatives of u and v :

$$\frac{\partial u}{\partial x}(0,0) = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x - 0} = 1$$

$$\frac{\partial u}{\partial y}(0,0) = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y - 0} = 0$$

$$\frac{\partial v}{\partial x}(0,0) = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x - 0} = 0$$

$$\frac{\partial v}{\partial y}(0,0) = \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y - 0} = 1$$

and the result follows. Another way to see this is as follows. For $y \in \mathbb{R} \setminus \{0\}$, we have $f(0,y) = f(iy) = iy$, thus

$$\frac{\partial f}{\partial y}(0,0) := \lim_{y \to 0} \frac{f(0,y) - f(0,0)}{y - 0} = i.$$

Similarly, we obtain

$$\frac{\partial f}{\partial x}(0,0) = 1.$$

Therefore,

$$\frac{\partial f}{\partial y}(0,0) = i = \frac{\partial f}{\partial x}(0,0)$$

i.e. the C-R equations are satisfied at the origin. But as we saw earlier, f is not complex differentiable at the origin.

Applications

1. Let $f : \mathbb{C} \to \mathbb{C}$ such that $f'(z) = 0$, $\forall z \in \mathbb{C}$. Then f is a constant function. This follows from the C-R equations and the Mean Value Theorem for two real variables and is left as an exercise.

2. If f is an entire function such that $f(z) \in \mathbb{R}, \forall z \in \mathbb{C}$, then f is a constant function. This follows immediately from the C-R equations for if we write $f = u + iv$, then $v \equiv 0$ and hence all partial derivatives of v are zero. Following the C-R equations, the partial derivatives of u are also zero.
   This tells us that the function $f(z) = |z|^2$, $z \in \mathbb{C}$ is not (complex) differentiable for $z \neq 0$, since if it where, then by the previous result, it would be a constant function, which is a contradiction.

3. Let $\Omega \subseteq \mathbb{C}$ a domain and let $f, g : \Omega \to \mathbb{C}$ holomorphic functions. Then:
   $\odot$ If $Ref = Reg$, then f and g differ by a constant. Indeed, write $f := u + iv_1$ and $g := u + iv_2$. Then,

$$f'(z) = \tfrac{\partial u}{\partial x} + i\tfrac{\partial v_1}{\partial x} \overset{C-R}{=} \tfrac{\partial u}{\partial x} - i\tfrac{\partial u}{\partial y}$$

and

$$g'(z) = \tfrac{\partial u}{\partial x} + i\tfrac{\partial v_2}{\partial x} \overset{C-R}{=} \tfrac{\partial u}{\partial x} - i\tfrac{\partial u}{\partial y}$$

Thus,

$$(f-g)'(z) = 0, \forall z \in \Omega \Longrightarrow f - g = c \in \mathbb{C}.$$

$\odot$ If $|f|$ is a constant, then f is also a constant. Indeed, suppose that $|f(z)| = c$, $\forall z \in \Omega$. Then

$$f = u + iv \Longrightarrow |f(z)|^2 = u^2(z) + v^2(z) = c^2, \forall z \in \Omega.$$

• Case 1: If $c = 0$, then $|f(z)| = 0 \Longleftrightarrow f(z) = 0$, $\forall z \in \Omega$.
• Case 2: If $c > 0$, then,

$$c^2 = u^2(z) + v^2(z) \Longrightarrow 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} \overset{C-R}{=} 2u\frac{\partial u}{\partial x} - 2v\frac{\partial u}{\partial y}$$

and

$$c^2 = u^2(z) + v^2(z) \Longrightarrow 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} \overset{C-R}{=} 2u\frac{\partial u}{\partial y} + 2v\frac{\partial u}{\partial x}.$$

Thus,

$$\left\{\begin{array}{l} u\dfrac{\partial u}{\partial x} - v\dfrac{\partial u}{\partial y} = 0 \\ u\dfrac{\partial u}{\partial y} + v\dfrac{\partial u}{\partial x} = 0 \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} u\dfrac{\partial u}{\partial x} - v\dfrac{\partial u}{\partial y} = 0 \\ v\dfrac{\partial u}{\partial x} + u\dfrac{\partial u}{\partial y} = 0 \end{array}\right\}$$

The last is a linear system of two equations of two unknowns, $X := \frac{\partial u}{\partial x}$ and $Y := \frac{\partial u}{\partial y}$. The discriminant of the system $= c^2 = u^2(z) + v^2(z) > 0 \Longrightarrow$ the system has only one solution, the trivial one, i.e.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0.$$

Also, by the *Cauchy − Riemann* equations, we have that

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0.$$

From all of the above, we obtain $f'(z) = 0 \Longrightarrow f \equiv c \in \mathbb{C}$.

Proposition 1.1. C-R and holomorphy

Let $f \in C^1(U)$, where $U \subseteq \mathbb{C}$ domain. Then, the Cauchy-Riemann equations (1.5) for $f = u + iv$ (in U) are necessary and sufficient conditions for f to be holomorphic.

Proof. (i)**Necessity** : For f be holomorphic, the

$$\begin{aligned} f'(z) &:= \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \\ &= \lim_{\substack{k\to 0 \\ \ell\to 0}} \frac{[u(x+k, y+\ell) + iv(x+k, y+\ell)] - [u(x,y) + iv(x,y)]}{k + i\ell} \end{aligned}$$

must exist and be independent of the direction in which h (equivalentlyk, ℓ) moves. We shall move in two main directions, perputating x and then y. This will illustrate the significance of C-R.

1. Suppose that $\ell = 0$ and $k \to 0$. Then the above limit becomes

$$\begin{aligned} & \lim_{k\to 0} \frac{[u(x+k, y) + iv(x+k, y)] - [u(x,y) + iv(x,y)]}{k} \\ = & \lim_{k\to 0} \left[\frac{u(x+k, y) - u(x,y)}{k} + i\frac{v(x+k, y) + iv(x,y)}{k}\right] \\ = & \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \end{aligned}$$

provided that the partial derivatives exist.

2. Suppose that $k = 0$ and $\ell \to 0$. Then the above limit becomes

$$\begin{aligned} & \lim_{\ell\to 0} \frac{[u(x, y+\ell) + iv(x, y+\ell)] - [u(x,y) + iv(x,y)]}{\ell} \\ = & \lim_{\ell\to 0} \left[\frac{u(x, y+\ell) - u(x,y)}{i\ell} + i\frac{v(x, y+\ell) + iv(x,y)}{\ell}\right] \\ = & \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}, \end{aligned}$$

provided that the partial derivatives exist. Thus, for f to be holomorphic, these two limits must coincide, therefore,

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

or equivalently

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

(ii)**Sufficiency** : Since $f \in C^1(U)$, the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are continuous. Then

$$\begin{aligned}
\Delta u &= u(x+\Delta x, y+\Delta y) - u(x,y) \\
&= [u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y)] + [u(x, y+\Delta y) - u(x,y)] \\
&= \left[\frac{\partial u}{\partial x} + \delta_1\right]\Delta x + \left[\frac{\partial u}{\partial y} + \eta_1\right]\Delta y \\
&= \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \delta_1\Delta x + \eta_1\Delta y,
\end{aligned}$$

where $\delta_1, \eta_1 \to 0$, while $\Delta x, \Delta y \to 0$. Similarly, since$f \in C^1(U)$, the partial derivatives $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are continuous. We have

$$\begin{aligned}
\Delta v &= v(x+\Delta x, y+\Delta y) - v(x,y) \\
&= [v(x+\Delta x, y+\Delta y) - v(x, y+\Delta y)] + [v(x, y+\Delta y) - v(x,y)] \\
&= \left(\frac{\partial v}{\partial x} + \delta_2\right)\Delta x + \left(\frac{\partial v}{\partial y} + \eta_2\right)\Delta y \\
&= \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \delta_2\Delta x + \eta_2\Delta y,
\end{aligned}$$

where $\delta_2, \eta_2 \to 0$, while $\Delta x, \Delta y \to 0$. Then

$$\Delta f(z) = \Delta u + i\Delta v = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y + \delta\Delta x + \eta\Delta y, \tag{1.6}$$

where $\delta = \delta_1 + i\delta_2 \to 0$ and $\eta = \eta_1 + i\eta_2 \to 0$ while $\Delta x, \Delta y \to 0$.
From the Cauchy-Riemann equations, (1.6) becomes

$$\begin{aligned}
\Delta f(z) &= \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\Delta x + \left[-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right]\Delta y + \delta\Delta x + \eta\Delta y \\
&= \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right](\Delta x + i\Delta y) + \delta\Delta x + \eta\Delta y \\
&= \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\Delta z + \delta\Delta x + \eta\Delta y \\
\Rightarrow \frac{\Delta f(z)}{\Delta z} &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \frac{\delta\Delta x + \eta\Delta y}{\Delta z} \\
\Rightarrow \lim_{\Delta z \to 0}\frac{\Delta f(z)}{\Delta z} &= \frac{df}{dz}(z) = f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.
\end{aligned}$$

Hence $f'(z)$ exists (for $z \in U$) and is unique. □

As you've noticed, the requirement that the function must be of class C^1 is found in all of the results in this section. We've seen that if for a function $f = u + iv$ defined in a domain D all partial derivatives (of order one) of u and v exist everywhere (in D) and are differentiable (as functions of two real variables) (in D) such that the C-R equations are satisfied for them, then f is holomorphic. An even weaker result is the one that supposes that all partial derivatives (of order one) of u and v exist everywhere (in D) and are continuous functions (in D), C-R equations are satisfied for them and f is continuous, then f must be holomorphic (in D).
At this point, one may wonder if weaker conditions for f may imposed so that holomorphicity occurs. Looman and Menchoff proved that the condition that partial derivatives of u and v must be continuous is superfluous, but C-R conditions are essential:

Theorem 1.4. Looman-Menchoff

Let $f = u + iv$ a **continuous** function in a domain D. If all partial derivatives (of order one) of u and v exist everywhere (in D) and satisfy the C-R equations everywhere in D, then f is holomorphic (in D).

Further discussion on this last result can be found in [10]. One can weaken the continuity hypothesis, and merely assume that f is bounded in D. This is a result, stated by Montel in 1913, and proved by G.P. Tolstov.

1.2. Differential operators and Holomorphic Functions

We've seen that the (complex) function $f(z) = |z|^2$, $z \in \mathbb{C}$ is (complex) differentiable only at the origin and thus it is not holomorphic there (i.e. there cannot exist an open neighborhood of $(0, 0)$ such that f is (complex) differentiable there). On the other hand, if we consider f in the real coordinates x and y, then it is everywhere (real) differentiable. This reflects the fact that $x^2 + y^2$ cannot be written as an expression not involving the term $\overline{z}$ (people calls this $'z\,barred'$). One can justify this geometrically by looking at the behaviour of the function $z \mapsto \overline{z}$. The class of holomorphic functions (and C^1) is quite interesting since we can validate holomorphicity 'conviniently' through a partial differentiable operator. This operator is expressed through the Cauchy-Riemann equations. (If you derive f with respect to $z\,bar$ and the result is 0, then f has no term involving $z\,bar$!)
In this section we study the complex derivative in more depth. For a review of the theory of multivariate Calculus and the theory of differential forms used here, refer to the appendix.

- Matrix representation

The Jacobian

Let $f = u + iv$ be a complex-valued function defined on an open subset U of $\mathbb{C}$. Then, under the usual identification of $\mathbb{C}$ with $\mathbb{R}^2$, we can consider the *Jacobian* of f:

$$Jf(z) = Jf(x + iy) = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix} \tag{1.7}$$

The following establishes the connection of the Jacobian with the complex derivative:

Proposition 2.1.

Let $f \in \mathcal{O}(U)$, where $U \subseteq \mathbb{C}$ open. Then

$$\det(Jf(z)) = |f'(z)|^2, \, \forall z \in U.$$

Proof. $\forall z = (x + iy) \in U$

$$\det(Jf(z)) \overset{(1.7)}{=} \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x} \overset{C-R}{=} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2.$$

But

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \Longrightarrow |f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

and the result follows. □

The Wirtinger derivative

We now establish the connection of the differential of a function with that of the notion of holomorphicity. It turns out that the check if a function is holomorphic or not can be done through a linear partial differential operator (see definition (2.1) in the appendix), namely the Wirtinger derivative. This operator is of great importance and ingenuity since it combines both complex and real coordinates.

We first define (complex) 1 and 2 forms. The reader may refer to the appendix for a brief expository of differential forms in general.

Definition 2.1.

Let $U \subseteq \mathbb{C}$. **A differential 1-form of type** (1,0) **and of degree** C^k $(k \in \mathbb{N})$ is a C^k function $\omega : U \to \mathcal{L}(\mathbb{C}, \mathbb{C})$, i.e. $\omega = Pdz$, where $P \in C^k_{\mathbb{C}}(U)$. The set of all differential 1-form of type (1,0) and of degree C^k is denoted by $\Omega^{(1,0)}_k(U)$. **A differential 1-form of type** (0,1) **and of degree** C^k $(k \in \mathbb{N})$ is a C^k function $\omega : U \to \overline{\mathcal{L}(\mathbb{C}, \mathbb{C})}$ i.e. $\omega = Pd\overline{z}$, where $P \in C^k_{\mathbb{C}}(U)$. The set of all differential 1-form of type (0,1) and of degree C^k is denoted by $\Omega^{(0,1)}_k(U)$ Thus:

$$\Omega^1_k(U) = \Omega^{(1,0)}_k(U) \oplus \Omega^{(0,1)}_k(U).$$

In other words, every $\omega \in \Omega^1_k(U)$ is written as $\omega = Pdz + Qd\overline{z}$, for $P, Q \in C^k_{\mathbb{C}}(U)$.
Similarly, a differential 2-form and of order C^K in U is a C^k function $\omega : U \to \mathcal{B}(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{C})$ and the space of all such forms is denoted with $\Omega^2_k(U)$. Each $\omega \in \Omega^2_k(U)$ is written as $\omega = Pdz \wedge d\overline{z}$, where $P \in C^k_{\mathbb{C}}(U)$.

Let $U \subseteq \mathbb{C}$ open and $f \in \mathcal{C}^1(U)$. Write $z = x + iy = (x, y)$ for $z \in U$. Then

$$dz = dx + idy \quad \text{and} \quad d\overline{z} = dx - idy.$$

Remember the connection between the x, y coordinates with the z, $\overline{z}$ ones:

$$x = \frac{z + \overline{z}}{2}, \qquad y = \frac{z - \overline{z}}{2i}.$$

These yield

$$dx = \frac{1}{2}(dz + d\overline{z}) \qquad dy = \frac{1}{2i}(dz - d\overline{z})$$

Then

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{1}{2}\frac{\partial f}{\partial x}(dz + d\overline{z}) + \frac{1}{2i}\frac{\partial f}{\partial x}(dz - d\overline{z}) \\ &= \frac{1}{2}\frac{\partial f}{\partial x}dz + \frac{1}{2}\frac{\partial f}{\partial x}d\overline{z} + \frac{1}{2i}\frac{\partial f}{\partial y}dz - \frac{1}{2i}\frac{\partial f}{\partial y}d\overline{z} \\ &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right)dz + \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right)d\overline{z} \\ &= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)dz + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)d\overline{z} \end{aligned}$$

Definition 2.2.

The linear partial operators (of order one) $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$ acting of C^1 functions f in $U \subseteq \mathbb{C}$ open as follows

$$\left(\frac{\partial f}{\partial z}\right)(f) = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \quad \text{and} \quad \left(\frac{\partial}{\partial \overline{z}}\right)(f) = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

are called the *Wirtinger derivatives of f with respect to z and $\overline{z}$.*

In this way, we've broken down the differential of f into two components $d = \partial + \overline{\partial}$, where $\partial f = \frac{\partial f}{\partial z}dz$ and $\overline{\partial} f = \frac{\partial f}{\partial \overline{z}}d\overline{z}$. ∂f corresponds to the (complex) linear part and the $\overline{\partial} f$ to the anti-linear one. Note that the Wirtinger derivatives should not be mistaken as the partial derivatives of f wrt z and $\overline{z}$ respectively, although, as we shall see later on, they behave the same as the derivatives of f wrt x and y. To be precise, if M is a complex differentiable manifold, $p \in M$ and z the coordinate vector around p, then the wirtinger operators (at the point p), i.e. the complexified vector fields at p, span the (complexified) tangent space $T_{\mathbb{C},p}(M)$ at p, which is dual to the (complexified) cotangent space at p spanned by the 1-forms dz and $d\overline{z}$. $T_{\mathbb{C},p}(M)$ is naturally defined as

$$T_{\mathbb{C},p}(M) := \mathbb{C} - span\left\{\frac{\partial}{\partial x}\Big|_p, \frac{\partial}{\partial y}\Big|_p\right\} = \mathbb{C} - span\left\{\frac{\partial}{\partial z}\Big|_p, \frac{\partial}{\partial \overline{z}}\Big|_p\right\}.$$

It follows that $T_{\mathbb{C},p}(M)$ is decomposed into two pieces:

$$T^{(1,0)}_{\mathbb{C},p}(M) = \mathbb{C} - span\left\{\frac{\partial}{\partial z}\Big|_p\right\} = \text{the holomorphic tangent space of } M \text{ at } p$$

and

$$T^{(0,1)}_{\mathbb{C},p}(M) = \mathbb{C} - span\left\{\frac{\partial}{\partial \overline{z}}\Big|_p\right\} = \text{the anti-holomorphic tangent space of } M \text{ at } p,$$

i.e.

$$T_{\mathbb{C},p}(M) = T^{(1,0)}_{\mathbb{C},p}(M) \oplus T^{(0,1)}_{\mathbb{C},p}(M).$$

$T^{(1,0)}_{\mathbb{C},p}(M)$ consists of all derivations that equal to zero on the anti-holomorphic functions, i.e. at $f : M \to \mathbb{C}$ with $\overline{f} \in \mathcal{O}(M)$ while $T^{(0,1)}_{\mathbb{C},p}(M)$ consists of all derivations that equal to zero on the holomorphic functions, i.e. at $f: M \to \mathbb{C}$ with $f \in \mathcal{O}(M)$. Also,

$$T^{(0,1)}_{\mathbb{C},p}(M) = \overline{T^{(1,0)}_{\mathbb{C},p}(M)}.$$

Now let $f: M \to N$ a differentiable function between two complex manifolds. Then the differential of f at each point $p \in M$ is

$$d_p f : T_{\mathbb{R},p}(M) \to T_{\mathbb{R},f(p)}(N)$$

and

$$\tilde{d}_p f : T^{(1,1)}_{\mathbb{C},p}(M) \to T^{(1,1)}_{\mathbb{C},f(p)}(N).$$

There not always exists a (linear) function

$$\tilde{d}_p f : T^{(1,0)}_{\mathbb{C},p}(M) \to T^{(1,0)}_{\mathbb{C},f(p)}(N).$$

In fact, a $f: M \to N$ is holomorphic $\Leftrightarrow$ $\tilde{d}_p f(T^{(1,0)}_{\mathbb{C},p}(M)) \subseteq T^{(1,0)}_{\mathbb{C},f(p)}(N)$, $\forall p \in M$.
Observe that the composition

$$T_{\mathbb{R},p}(M) \to T^{(1,1)}_{\mathbb{C},p}(M) \to T^{(1,0)}_{\mathbb{C},p}(M)$$

is a $\mathbb{R}-$ linear isomorphism, which allows us to do geometry in the z coordinate: if $z(t) = (x(t), y(t))$, then

$$z'(t) = x'(t)\frac{\partial}{\partial x}\Big|_p + y'(t)\frac{\partial}{\partial y}\Big|_p \in T_{\mathbb{R},p}(\mathbb{C})$$

or

$$z'(t)\frac{\partial}{\partial z}\Big|_p \in T^{(1,0)}_{\mathbb{C},p}(\mathbb{C}).$$

The following comes completely naturally:

Proposition 2.2.

Let $f \in C^1(U)$ where $U \subseteq \mathbb{C}$ open. Then

$$\frac{\partial f}{\partial \overline{z}} = 0 \Longleftrightarrow f \in \mathcal{O}(U).$$

Proof.

$$\frac{\partial f}{\partial \overline{z}} = 0 \Longleftrightarrow \frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y} = 0 \Longleftrightarrow \frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y} \Longleftrightarrow \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}.$$

Write $f = u + iv$. Then the above equations become

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) \Longleftrightarrow \frac{\partial u}{\partial x} = -i^2\frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} \quad \text{and} \quad i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y},$$

or equivalently

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

which are the C-R equations for f on U. ☐

An important thing to remember here is that if $f : U \to \mathbb{C}$ is a C^1 function, where $U \subseteq \mathbb{C}$ open satisfying the Cauchy-Riemann equations, then

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

(on U). Also if we write $f = u + iv$ then the Wirtinger derivatives of f are

$$\left(\frac{\partial}{\partial z}\right)(f) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{1.8}$$

and

$$\left(\frac{\partial}{\partial \overline{z}}\right)(f) = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \tag{1.9}$$

respectively.

It is also easy to verify that the following equality holds:

$$d\overline{z} \wedge dz = 2idx \wedge dy$$

or (due to the antisymmetry of the wedge product)

$$dz \wedge d\overline{z} = -2idx \wedge dy.$$

this reflects the fact that for $z = (x, y) \in U \subseteq \mathbb{C}$ domain, we have

$$\Omega^2_{(x,y)}\mathbb{C} = span\{dx \wedge dy\} = span\{dz \wedge d\overline{z}\}.$$

Finally, we give calculations for the pull-backs of differential forms on the complex plane.

Case 1: $U_1, U_2 \subseteq \mathbb{R}$ open

Consider a function $g : U_2 \to \mathbb{C}$ (a 0-form). Then, the pullback $f^*(g) : U_1 \to \mathbb{C}$ of g through f is defined

$$f^*(g) := g \circ f.$$

If ω is a 1-form with coefficients defined in U_2, (i.e.. $\omega = gdt$, for a C^∞ function g), then its pullback through f is defined as

$$f^*(\omega) := (g \circ f)df = (g \circ f)f'dt.$$

Case 2: $U_1 \subseteq \mathbb{R}$, $U_2 \subseteq \mathbb{C}$ open

Consider a function $g : U_2 \to \mathbb{C}$ (a 0-form). Then, the pullback $f^*(g) : U_1 \to \mathbb{C}$ of g through f is defined in the same way as before:

$$f^*(g) := g \circ f.$$

If ω is a 1-form with coefficients functions defined on U_2, (: $\omega = Pdx + Qdy$, for some C^∞ functions P, Q), then the pullback of f is defined as

$$f^*(\omega) := (P \circ f)d(x \circ f) + (Q \circ f)d(y \circ f).$$

If $f = (f_1, f_2) = f_1 + if_2$, then the above becomes

$$f^*(\omega) := [(P \circ f)f_1' + (Q \circ f)f_2']dt.$$

The 1-differential form ω can also be expressed using the basis $\{dz,\ d\overline{z}\}$: Write $\omega = Adz + Bd\overline{z}$. Thus, we can derive the pullback of ω through f :

$$\begin{aligned} f^*(\omega) &= (A \circ f)df + (B \circ f)d\overline{f} \\ &= (A \circ f)(f' dt) + (B \circ f)(\overline{f}' dt) \\ &= (A \circ f)\left(\frac{df_1}{dt}dt + i\frac{df_2}{dt}dt\right) + (B \circ f)\left(\frac{df_1}{dt}dt - i\frac{df_2}{dt}dt\right) \\ &= (A \circ f)\left(\frac{df_1}{dt} + i\frac{df_2}{dt}\right)dt + (B \circ f)\left(\frac{df_1}{dt} - i\frac{df_2}{dt}\right)dt. \end{aligned}$$

Case 3: $U_1, U_2 \subseteq \mathbb{C}$ **open**

If $g : U_2 \to \mathbb{C}$ is a function, then its pullback $f^*(g) : U_1 \to \mathbb{C}$ through f is defined as before:

$$f^*(g) := g \circ f.$$

If ω is a 1-form with coefficients functions defined in U_2, (:$\omega = Pdx + Qdy$, for some C^∞ functions P, Q), then

$$\begin{aligned} f^*(\omega) &:= (P \circ f)d(x \circ f) + (Q \circ f)d(y \circ f) = (P \circ f)df_1 + (Q \circ f)df_2 \\ &= (P \circ f)\left[\frac{\partial f_1}{\partial \xi}d\xi + \frac{\partial f_1}{\partial \eta}d\eta\right] + (Q \circ f)\left[\frac{\partial f_2}{\partial \xi}d\xi + \frac{\partial f_2}{\partial \eta}d\eta\right], \\ &= \left((P \circ f)\frac{\partial f_1}{\partial \xi} + (Q \circ f)\frac{\partial f_2}{\partial \xi}\right)d\xi + \left((P \circ f)\frac{\partial f_1}{\partial \eta} + (Q \circ f)\frac{\partial f_2}{\partial \eta}\right)d\eta, \end{aligned}$$

where $f = (f_1, f_2) = f_1 + if_2,\ z = x + iy = (x, y) \in\ U_2$ and $\zeta = \xi + i\eta = (\xi, \eta) \in\ U_1$. The 1-differential form ω can also be expressed using the basis $\{dz,\ d\overline{z}\}$: (exercise-write $\omega = Adz + Bd\overline{z}$) If $\omega = Pdx \wedge dy$, then

$$\begin{aligned} f^*(\omega) &= (P \circ f)f^*(dx) \wedge f^*(dy) = (P \circ f)d(x \circ f) \wedge d(y \circ f) \\ &= (P \circ f)df_1 \wedge df_2 = (P \circ f)\left[\left(\frac{\partial f_1}{\partial \xi}d\xi + \frac{\partial f_1}{\partial \eta}d\eta\right) \wedge \left(\frac{\partial f_2}{\partial \xi}d\xi + \frac{\partial f_2}{\partial \eta}d\eta\right)\right] \\ &= (P \circ f)\left[\frac{\partial f_1}{d\xi}\frac{\partial f_2}{d\eta} - \frac{\partial f_1}{d\eta}\frac{\partial f_2}{d\xi}\right]d\xi \wedge d\eta \\ &= (P \circ f)|J(f)|d\xi \wedge d\eta, \end{aligned}$$

Thus

$$\begin{aligned}
f^*(\omega) &= (A\circ f)df\wedge d\overline{f}\\
&= (A\circ f)\left(\frac{\partial f}{\partial \zeta}d\zeta+\frac{\partial f}{\partial \overline{\zeta}}d\overline{\zeta}\right)\wedge\left(\frac{\partial \overline{f}}{\partial \zeta}d\zeta+\frac{\partial \overline{f}}{\partial \overline{\zeta}}d\overline{\zeta}\right)\\
&= (A\circ f)\left[\left(\frac{\partial f_1}{\partial \zeta}+i\frac{\partial f_2}{\partial \zeta}\right)d\zeta+\left(\frac{\partial f_1}{\partial \overline{\zeta}}+i\frac{\partial f_2}{\partial \overline{\zeta}}\right)d\overline{\zeta}\right]\wedge\\
&\wedge\left[\left(\frac{\partial f_1}{\partial \zeta}-i\frac{\partial f_2}{\partial \zeta}\right)d\zeta+\left(\frac{\partial f_1}{\partial \overline{\zeta}}-i\frac{\partial f_2}{\partial \overline{\zeta}}\right)d\overline{\zeta}\right]\\
&= (A\circ f)\left[\left(\frac{\partial f_1}{\partial \zeta}\right)^2-\left(\frac{\partial f_1}{\partial \overline{\zeta}}\right)^2+\left(\frac{\partial f_1}{\partial \zeta}\right)^2+\left(\frac{\partial f_2}{\partial \overline{\zeta}}\right)^2\right]d\zeta\wedge d\overline{\zeta}\\
&= (A\circ f)\left(\left|\frac{\partial f}{\partial \zeta}\right|^2-\left|\frac{\partial f}{\partial \overline{\zeta}}\right|^2\right)d\zeta\wedge d\overline{\zeta}
\end{aligned}$$

EXAMPLES and APPLICATIONS

Example 1

Elementary calculations yield

$$\frac{\partial z}{\partial z} = 1 = \frac{\partial \overline{z}}{\partial \overline{z}} \quad \text{and} \quad \frac{\partial z}{\partial \overline{z}} = 0 = \frac{\partial \overline{z}}{\partial z}.$$

Also

$$\overline{\frac{\partial(|z|^2)}{\partial z}} = z = \frac{\partial(|z|^2)}{\partial \overline{z}}.$$

We prove only the last one: write $z = x + iy$. Then $\overline{z} = x - iy$. Thus,

$$\begin{aligned}\frac{\partial(|z|^2)}{\partial z} &= \frac{1}{2}\left(\frac{\partial(x^2+y^2)}{\partial x} - i\frac{\partial(x^2+y^2)}{\partial y}\right) \\ &= \frac{1}{2}(2x - 2iy) = x - iy = \overline{z}.\end{aligned}$$

Example 2

For $z \neq 0$, then

$$\frac{\partial(z^{-1})}{\partial \overline{z}} = 0.$$

Indeed, write $z = x + iy$. Then, as we've already seen,

$$\frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}.$$

Thus,

$$\frac{\partial(z^{-1})}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial(z^{-1})}{\partial x} + i\frac{\partial(z^{-1})}{\partial y}\right).$$

Now, after a few computations we obtain

$$\frac{\partial(z^{-1})}{\partial x} = \frac{y^2 - x^2}{(x^2+y^2)^2} + i\frac{2xy}{(x^2+y^2)^2}$$

and

$$\frac{\partial(z^{-1})}{\partial y} = -\frac{2xy}{(x^2+y^2)^2} + i\frac{y^2-x^2}{(x^2+y^2)^2}.$$

Hence

$$\frac{\partial(z^{-1})}{\partial x} = -i\frac{\partial(z^{-1})}{\partial y}$$

and we are done.

Example 3

Let $f = u + iv$ a C^1 function defined on an open subset U of $\mathbb{C}$. We will calculate

$$\frac{\partial \overline{f}}{\partial z} \quad \text{and} \quad \frac{\partial \overline{f}}{\partial \overline{z}}$$

in terms of the x, y coordinates.

Although these are elementary calculations for the one familiar with the Theory of Differential Operators and/or Complex Analysis, the explicit calculations carried out here will benefit the reader undertaking a first course in Complex Analysis.
$\overline{f} = u - iv$. Let $z = (x + iy) \in U$. Then

$$\begin{aligned}\frac{\partial \overline{f}}{\partial z} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial x}\right) \\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x} - i\frac{\partial u}{\partial y} + i^2\frac{\partial v}{\partial y}\right) \\ &= \frac{1}{2}\left[\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) - i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]\end{aligned}$$

and similarly

$$\frac{\partial \overline{f}}{\partial \overline{z}} = \frac{1}{2}\left[\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)\right]$$

By the above calculations and formulas (1.8) and (1.9) we immediately see that

$$\frac{\partial \overline{f}}{\partial \overline{z}} = \overline{\frac{\partial f}{\partial z}} \quad \text{and} \quad \frac{\partial \overline{f}}{\partial z} = \overline{\frac{\partial f}{\partial \overline{z}}}$$

APPLICATION: Change of Basis/The complex chain rule

With the help of the chain rule (in 2 real variables), calculations done in the last example immediately yield the corresponding rule for changing the basis $\{\partial/\partial x, \partial/\partial y\}$ to the basis $\{\partial/\partial z, \partial/\partial \overline{z}\}$. More precisely, by complexifying the tangent bundle $T_p\mathbb{R}^2$ (where $p = (x, y) \in Domain(f)$ and f is regarded as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$) the Jacobian matrix representing $d_{p,\mathbb{C}}(f)$ in this base is

$$J_{p,\mathbb{C}}(f) = \begin{pmatrix} \dfrac{\partial f}{\partial z}\Big|_p & \dfrac{\partial f}{\partial \bar{z}}\Big|_p \\ \dfrac{\partial \bar{f}}{\partial z}\Big|_p & \dfrac{\partial \bar{f}}{\partial \bar{z}}\Big|_p \end{pmatrix}$$

The change of basis matrix $Q \in \mathcal{M}_{2\times 2}(\mathbb{C})$ is

$$Q = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

and thus

$$J_{p,\mathbb{C}}(f) = Q \cdot \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix} \cdot Q^{-1}$$

This also tells us that the Wirtinger derivatives behave as the derivatives with respect to x and y.
Finally, if g is a suitable function, then by the chain rule, $d_{p,\mathbb{C}}(f \circ g) = d_{w=g(p),\mathbb{C}}(f) \circ d_{p,\mathbb{C}}(g)$, which in matrix form is translated into

$$J_{p,\mathbb{C}}(f \circ g) = J_{g(p),\mathbb{C}}(f) \cdot J_{p,\mathbb{C}}(g),$$

that is:

$$\begin{pmatrix} \dfrac{\partial (f\circ g)}{\partial z}\Big|_p & \dfrac{\partial (f\circ g)}{\partial \bar{z}}\Big|_p \\ \dfrac{\partial \overline{(f\circ g)}}{\partial z}\Big|_p & \dfrac{\partial \overline{(f\circ g)}}{\partial \overline{z}}\Big|_p \end{pmatrix} = \begin{pmatrix} \dfrac{\partial f}{\partial z}\Big|_{w=g(p)} & \dfrac{\partial f}{\partial \bar{z}}\Big|_{w=g(p)} \\ \dfrac{\partial \bar{f}}{\partial z}\Big|_{w=g(p)} & \dfrac{\partial \bar{f}}{\partial \bar{z}}\Big|_{w=g(p)} \end{pmatrix} \cdot \begin{pmatrix} \dfrac{\partial g}{\partial z}\Big|_p & \dfrac{\partial g}{\partial \bar{z}}\Big|_p \\ \dfrac{\partial \bar{g}}{\partial z}\Big|_p & \dfrac{\partial \bar{g}}{\partial \bar{z}}\Big|_p \end{pmatrix}$$

We can remember the Complex Chain rule as follows:

$$\frac{\partial}{\partial z}(f \circ g) = \frac{\partial f}{\partial w}\Big|_{w=g(z)} \cdot \frac{\partial g}{\partial z} + \frac{\partial f}{\partial \bar{w}}\Big|_{w=g(z)} \cdot \frac{\partial \bar{g}}{\partial z} \tag{1.10}$$

where the function f is of course regarded as a function in both z and $\overline{z}$.

1.3. Harmonic conjugates and primitive functions

- We leave as an exercise for the reader to prove that if $u \in C^2(U)$, where $U \subseteq \mathbb{C}$ open, then

$$4\frac{\partial^2 u}{\partial z \partial \overline{z}} = \Delta u = 4\frac{\partial^2 u}{\partial \overline{z} \partial z}.$$

Lets overview of what we have seen so far. A continuously differentiable function $f : U \to \mathbb{C}$ where $U \subseteq \mathbb{C}$ open is holomorphic iff it satisfies the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1.11}$$

where $u = Re(f)$ and $v = Im(f)$. In other words, the hypotheses that both partial derivatives of f with respect to x and y are continuous functions is necessary and sufficient for f to be holomorphic. We've defined the Wirtinger partial differential operators:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \tag{1.12}$$

and

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \tag{1.13}$$

acting on C^1 functions:

$$f \mapsto \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \tag{1.14}$$

and

$$f \mapsto \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \tag{1.15}$$

respectively. If we write $f = u + iv$ then (1.14) and (1.15) become

$$\left(\frac{\partial}{\partial z}\right)(f) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{1.16}$$

and

$$\left(\frac{\partial}{\partial \overline{z}}\right)(f) = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \tag{1.17}$$

respectively. These operators are an important tool in establishing relations of considering a complex function f as a function of two real variables. The next result follows immediately:

Lemma 3.1. Let $f : U \to \mathbb{C}$ a C^1 function, where $U \subseteq \mathbb{C}$ open. If f satisfies the Cauchy-Riemann equations (1.11), then

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

(on U).

Proof.

$$\begin{aligned}\frac{\partial f}{\partial z} &\overset{(1.16),(1.11)}{=} \frac{1}{2}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)+\frac{i}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\\ &= \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial f}{\partial x}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial f}{\partial z} &\overset{(1.17),(1.11)}{=} \frac{1}{2}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial y}\right)+\frac{i}{2}\left(-\frac{\partial u}{\partial y}-\frac{\partial u}{\partial y}\right)\\ &= \frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}=-i\left(-\frac{1}{i}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial y}\right)\\ &= -i\left(i\frac{\partial v}{\partial y}+\frac{\partial u}{\partial y}\right)=-i\frac{\partial f}{\partial y}\end{aligned}$$

□

Now, if $u, v \in C^2(U)$ that satisfy the C-R equations, using the mixed partial derivatives property (Clairaut-Schwarz Theorem) we get

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)=-\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=-\frac{\partial^2 u}{\partial y^2}.$$

Thus,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \tag{1.18}$$

Similarly,

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \tag{1.19}$$

Definition 3.1 (The Laplace operator)**.** Let $U \subseteq \mathbb{C}$ open. The operator

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

acting on C^2 functions in U is called the *Laplace operator*. If $u \in C^2(U)$ satisfies $\Delta u = 0$, then u is called *harmonic*.

Since for an holomorphic function $f = u + iv$ the partial derivatives of all orders exist and are continuous functions, we immediately have that u and v are harmonic functions. In other words, real and imaginary parts of a holomorphic function are harmonic functions (see (1.18) and (1.19)). In this section, we will deal with the opposite of this last fact: given a harmonic function u (of two real variables), whether there exist a harmonic function v such that the function $f = u + iv$ is holomorphic. Since the partial derivatives of v with respect to x and y are fully determined by the corresponding partial derivatives of the given function u (since the C-R equations must be valid), the problem is equivalent into solving an antiderivative problem in two (real) dimensions:

$$do\, F, G \in C^1(U)\, exist\, such\, that\, \frac{\partial v}{\partial x}=F \quad and \quad \frac{\partial v}{\partial y}=G?$$

Theorem 3.1. Let $U \subsetneqq \mathbb{C}$ be a simply connected region and let $u : U \to \mathbb{R}^2$ such that $\Delta u = 0$. Then there exists a function $v : U \to \mathbb{R}^2$ such that the function $f = u + iv$ is holomorphic (on U). In this case, v is called the harmonic conjugate to u.

To connect the idea behind the proof with the discussion we did at the beginning of this paragraph, suppose that u is a given harmonic function. We want to construct a harmonic conjugate of u, that is a function v that satisfies the C-R equations $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$. Then

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy.$$

But dv is an exact differential. Indeed, set $P(x,y) = -\frac{\partial u}{\partial y}$ and $Q(x,y) = -\frac{\partial u}{\partial x}$. Then $\frac{\partial P}{\partial y} = -\frac{\partial^2 u}{\partial y^2}$ and $\frac{\partial Q}{\partial x} = \frac{\partial^2 u}{\partial x^2}$ and since $\Delta u = 0$, we have that

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = \Delta u = 0$$

and the result follows. Thus, constructing an antiderivative is done in the usual way as in multivariate calculus. In accordance to this, lets study the case of the disc:

Lemma 3.2.

Every harmonic function u in a disc has a harmonic conjugate.

Proof. We can suppose without loss of generality, that the disc is $\Delta(0,1)$ and construct the harmonic conjugate v of u. The C-R equations must be satisfied:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1.20}$$

Let v be the function defined by

$$v(x,y) = \int_0^y \frac{\partial u}{\partial x}(x,t)\,dt + c(x), \ \forall z = (x,y) \in \Delta(0,1)$$

where c is a smooth function (in x). Notice that v is well defined. Let $x \in D$ fixed. By a well known Theorem that allows us to interchange integration and differentiation, we have

$$\begin{aligned}
\frac{\partial v}{\partial x} &= \int_0^y \frac{\partial^2 u}{\partial y^2}(x,t)\,dt + c'(x) \\
(\textit{since}\ \Delta u = 0) &= -\int_0^y \frac{\partial^2 u}{\partial y^2}(x,t)\,dt + c'(x) \\
&= -\left[\frac{\partial u}{\partial y}(x,t)\right]_{t=0}^{y} + c'(x) \\
&= \frac{\partial u}{\partial y}(x,0) - \frac{\partial u}{\partial y}(x,y) + c'(x)
\end{aligned}$$

But then,

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Leftrightarrow c'(x) = -\frac{\partial v}{\partial y}(x,0)$$

and thus

$$c(x) = -\int_0^x \frac{\partial u}{\partial y}(t, 0)\, dt + c$$

for some real constant c and then

$$v(x, y) = \int_0^y \frac{\partial u}{\partial x}(x, t)\, dt - \int_0^x \frac{\partial u}{\partial y}(t, 0)\, dt + c.$$

□

Lemma 3.3. -Properties of harmonic functions and harmonic conjugates

Let D be a domain in the complex plain.

1. The sum of two harmonic functions (in D) is harmonic.
2. If v is a harmonic conjugate of u (in D), then every harmonic conjugate w of u differs with v by a real constant.
3. If v is a harmonic conjugate of u (in D), then $-u$ is a harmonic conjugate of v.
4. If v is a harmonic conjugate of u (in D), then $u \cdot v$ is harmonic.

Proof.

1. Let u and v two harmonic functions (in D). Then $\Delta(u + v) = \Delta u + \Delta v = 0$ and the result follows.
2. Let $v, \tilde{v}$ be two harmonic conjugates of u (in D). Define the function $q = v - \tilde{v}$. Then the C-R equations (1.5) yield
$$\frac{\partial q}{\partial x} = \frac{\partial \tilde{v}}{\partial x} - \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} - \left(-\frac{\partial u}{\partial y}\right) = 0$$
and
$$\frac{\partial q}{\partial y} = \frac{\partial \tilde{v}}{\partial y} - \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} - \frac{\partial u}{\partial x} = 0$$
and thus, since D is a domain, Theorem (1.2) yields q is constant.
3. If v is a harmonic conjugate of u (in D), then the function $f = u + iv$ is holomorphic in D. Then the function g defined by
$$g(z) = -f \cdot f(z) = -i(u + iv) = v - iu$$
is also holomorphic in D (use the C-R equations on g). Thus, $-u$ is a harmonic conjugate of v.
4. Use the same argument as in the previous result for the function $g = \frac{1}{2}f^2 = \frac{1}{2}(u^2 - v^2) + (uv)i$, for then g is holomorphic in D, $uv = \frac{1}{2}Im(f)$ and $Re(g)$ is harmonic.

□

Note that the product two harmonic functions (in D) is not always harmonic. Consider for example the functions u and v defined by $u(x, y) = v(x, y) = x$ or $u(x, y) = x$ and $v(x, y) = x^2 + y^2$.

An important example

The following example illustrates the importance of the condition that U is simply connected.

Let
$$u(x,y) = \frac{1}{2}\log(x^2+y^2).$$
Then u is indeed harmonic:
$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} \Leftrightarrow \frac{\partial^2 u}{\partial x^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$
and due to symmetry,
$$\frac{\partial^2 u}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}.$$
Thus $\Delta u = 0$. There can be no $f \in \mathcal{O}(\mathbb{C} \setminus \{0\})$ such that $Re(f) = u$ as it is not possible to extend u on a bigger than $(-\infty, 0]$ set. Indeed, suppose that there existed $f \in \mathcal{O}(\mathbb{C} \setminus \{0\})$, $f = u + iv$. Then to find the harmonic conjugate v of u, we use the C-R equations:
$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} \Longleftrightarrow v(x,y) = \int \frac{x}{x^2+y^2}dy = \arctan\left(\frac{y}{x}\right) + c(x).$$
Then
$$\frac{\partial v}{\partial x} = -\frac{y}{x^2+y^2} + c'(x).$$
Thus,
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Longleftrightarrow \frac{y}{x^2+y^2} = \frac{y}{x^2+y^2} + c'(x) \Longleftrightarrow c'(x) = 0 \Longleftrightarrow c(x) = c = constant.$$
Therefore
$$v(x,y) = \arctan\left(\frac{y}{x}\right) + c$$
and
$$f(z) = \frac{1}{2}\log(x^2+y^2) + i\arctan\left(\frac{y}{x}\right) + ic = \log|z| + i\text{Arg}(z) + ic.$$
If g is the logarithmic function:
$$g(z) := \log z = \log|z| + i\text{Arg}(z),\ z \in \mathbb{C} \setminus (-\infty, 0],$$
then $Re(f) = Re(g)$ and then (see exercises) $f = g + c$ on $\mathbb{C} \setminus (-\infty, 0]$ and thus g is extended as a holomorphic function on $\mathbb{C} \setminus (-\infty, 0]$. But that is not possible since
$$\lim_{(x,y)\to(x_0,0+)} \text{Arg}(x,y) = A \neq \lim_{(x,y)\to(x_0,0-)} \text{Arg}(x,y) = A + 2\pi.$$
This example illustrates what we've mentioned before: if the domain of u is not simply connected, then u can be completed to an analytic function $f = u + iv$ which is not single valued. As we shall see later on, behind this lies the problem of finding a primitive function. In this specific example, one can deduce that the function $z \mapsto \frac{1}{z}$ has a primitive function on $\mathbb{C}\setminus(-\infty, 0]$ (which is the logarithmic function) but no primitive on $\mathbb{C} \setminus \{0\}$.

Bibliography

[1] John B. Conway - *Functions of One Complex Variable I,* Second Edition, 1978 Springer-Verlag New York

[2] Gennadi Henkin, Jürgen Leiterer - *Theory of Functions on Complex Manifolds,* 1984 Birkhäuser Verlag

[3] L. Hörmander - *An Introduction to COMPLEX ANALYSIS in Several Variables* 1973, North Holland, Amsterdam

[4] Steven G. Krantz - *Function Theory of Several Complex Variables,* 2001 Corrected Printing, AMS CHELSEA PUBLISHING

[5] James Ward Brown, Ruel Vance Churchill - *Complex Variables and Applications, 8th Edition*McGraw-Hill Higher Education, 2009

[6] Robert E. Greene and Steven G. Krantz - *Function Theory of One Complex Variable: Third Edition,* 2006 Graduate Studies in Mathematics, vol. 40, AMS

[7] R. Michael Range - *Holomorphic Functions and Integral Representations in Several Complex Variables,* Springer Verlag, Berlin, 1998 Corrected Printing

[8] Walter Rudin - *Real and Complex Analysis,* Third Edition, 1987 McGRAW HILL INTERNATIONAL EDITIONS

[9] E. H. Connell and P. Porcelli - *A proof of the power series expansion without Cauchy's formula,* Bull. Amer. Math. Soc. 67 (1961), 177-181

[10] J. D. Gray and S. A. Morris - *When is a Function that Satisfies the Cauchy-Riemann Equations Analytic?,* The American Mathematical Monthly Vol. 85, No. 4 (Apr., 1978), pp. 246-256

[11] W. Wirtinger - *Zur formalen Theorie der Funktionen von mehr komplexen Veränderlichen,* Mathematische Annalen, 97:357-375, 1926