Ασκήσεις (Ο κανόνας της Αλυσίδας)

Ο κανόνας της Αλυσίδας

Έστω η συνάρτηση $F : \mathbb{R}^2 \to \mathbb{R}$. Θεωρούμε τη συνάρτηση $g(u,v) = F(x(u,v), y(u,v))$. Από τον κανόνα της αλυσίδας,

$$\boxed{\frac{\partial g}{\partial u} = \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial u}} \quad \text{και} \quad \boxed{\frac{\partial g}{\partial v} = \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial v}}$$

Αν $g : \mathbb{R}^2 \to \mathbb{R}$ συνάρτηση και $x = x(t)$, $y = y(t)$, τότε για τη συνάρτηση $h(t) = g(x(t), y(t))$,

$$\boxed{h'(t) = x'(t)\frac{\partial g}{\partial x} + y'(t)\frac{\partial g}{\partial y}}$$

δηλ. (με τη μορφή εσωτερικού γινομένου)

http://ioakimioannis.com

$$h'(t) = (x'(t), y'(t)) \cdot \begin{pmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \end{pmatrix}.$$

1. Έστω $F : \mathbb{R}^2 \to \mathbb{R}$ παραγωγίσιμη συνάρτηση. Θεωρούμε τη συνάρτηση $g(u,v) = F(x(u,v), y(u,v))$, όπου $x(u,v) = u^2 - v^2$ και $y(u,v) = v^2 - u^2$. Να δείξετε ότι

$$v\frac{\partial g}{\partial u} + u\frac{\partial g}{\partial v} = 0$$

Απάντηση

$$\begin{aligned}\frac{\partial g}{\partial u} &= \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial u} = 2u\cdot\frac{\partial F}{\partial x} - 2u\cdot\frac{\partial F}{\partial y} \\ &= 2u\left(\frac{\partial F}{\partial x} - \frac{\partial F}{\partial y}\right)\end{aligned}$$

και

$$\begin{aligned}\frac{\partial g}{\partial v} &= \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial v} = -2v\cdot\frac{\partial F}{\partial x} + 2v\cdot\frac{\partial F}{\partial y} \\ &= -2v\left(\frac{\partial F}{\partial x} - \frac{\partial F}{\partial y}\right)\end{aligned}$$

Άρα,

$$v\frac{\partial g}{\partial u} + u\frac{\partial g}{\partial v} = 2uv\left(\frac{\partial F}{\partial x} - \frac{\partial F}{\partial y}\right) - 2uv\left(\frac{\partial F}{\partial x} - \frac{\partial F}{\partial y}\right) = 0.$$

2. Έστω η C^1 συνάρτηση $F : \mathbb{R}^2 \to \mathbb{R}$. Θεωρούμε τη συνάρτηση $H : \mathbb{R}^2 \to \mathbb{R}$ με

$$H(u,v) = -v + F(u^2 - v^3, e^{-u}).$$

Να βρείτε τις $\frac{\partial H}{\partial u}$ και $\frac{\partial H}{\partial v}$.

Απάντηση

$$\frac{\partial H}{\partial u} = 0 + \frac{\partial F}{\partial u} = \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial u} = 2u\cdot\frac{\partial F}{\partial x} - e^{-u}\cdot\frac{\partial F}{\partial y}$$

και

$$\begin{aligned}\frac{\partial H}{\partial v} &= -1+\frac{\partial F}{\partial v} = \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial v} = 1-3v^2\cdot\frac{\partial F}{\partial x}+0\\ &= 1-3v^2\cdot\frac{\partial F}{\partial x}\end{aligned}$$

(3) Έστω η συνάρτηση $F:\mathbb{R}^2\to\mathbb{R}$ και οι συναρτήσεις $u(x,y)=x\cos\lambda+y\sin\lambda$ και $v(x,y)=x\sin\lambda-y\cos\lambda$, όπου λ πραγματική σταθερά. Αν $g(x,y)=F(u(x,y),v(x,y))$, να δείξετε ότι

$$\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2=\left(\frac{\partial F}{\partial u}\right)^2+\left(\frac{\partial F}{\partial v}\right)^2$$

Απάντηση

$$\frac{\partial g}{\partial x} = \frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial F}{\partial v}\cdot\frac{\partial v}{\partial x} = \cos\lambda\cdot\frac{\partial F}{\partial u}+\sin\lambda\cdot\frac{\partial F}{\partial v}$$

και

$$\frac{\partial g}{\partial y} = \frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial F}{\partial v}\cdot\frac{\partial v}{\partial y} = \sin\lambda\cdot\frac{\partial F}{\partial u}-\cos\lambda\cdot\frac{\partial F}{\partial v}$$

Τότε,

$$\begin{aligned}\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2 &= \left(\cos\lambda\cdot\frac{\partial F}{\partial u}+\sin\lambda\cdot\frac{\partial F}{\partial v}\right)^2+\left(\sin\lambda\cdot\frac{\partial F}{\partial u}-\cos\lambda\cdot\frac{\partial F}{\partial v}\right)^2\\ &= \cos^2\lambda\left(\frac{\partial F}{\partial u}\right)^2+2\cos\lambda\sin\lambda\frac{\partial F}{\partial u}\frac{\partial F}{\partial v}+\sin^2\lambda\left(\frac{\partial F}{\partial v}\right)^2\\ &\quad+\sin^2\lambda\left(\frac{\partial F}{\partial u}\right)^2-2\cos\lambda\sin\lambda\frac{\partial F}{\partial u}\frac{\partial F}{\partial v}+\cos^2\lambda\left(\frac{\partial F}{\partial v}\right)^2\\ &= (\sin^2\lambda+\cos^2\lambda)\left(\frac{\partial F}{\partial u}\right)^2+(sin^2\lambda+\cos^2\lambda)\left(\frac{\partial F}{\partial v}\right)^2\\ &= \left(\frac{\partial F}{\partial u}\right)^2+\left(\frac{\partial F}{\partial v}\right)^2\end{aligned}$$

(4) Έστω $F:\mathbb{R}^2\to\mathbb{R}$ μια C^1 συνάρτηση. Θέτουμε $x=u+v$ και $y=uv$ και θεωρούμε τη συνάρτηση $h(u,v)=F(x(u,v),y(u,v))$. Να δείξετε ότι

$$\frac{\partial F}{\partial x}=\frac{1}{u-v}\left(u\frac{\partial h}{\partial u}-v\frac{\partial h}{\partial v}\right)$$

και

$$\frac{\partial F}{\partial y}=\frac{1}{u-v}\left(\frac{\partial h}{\partial v}-\frac{\partial h}{\partial u}\right)$$

Απάντηση

$$\frac{\partial h}{\partial u} = \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}$$

και

$$\frac{\partial h}{\partial v} = \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{\partial F}{\partial x}+u\frac{\partial F}{\partial y}$$

Πρέπει να λύσουμε το σύστημα

$$\begin{cases} \dfrac{\partial F}{\partial x} + v\dfrac{\partial F}{\partial y} = \dfrac{\partial h}{\partial u} \\ \\ \dfrac{\partial F}{\partial x} + u\dfrac{\partial F}{\partial y} = \dfrac{\partial h}{\partial v} \end{cases}$$

(άγνωστοι είναι οι $\frac{\partial F}{\partial x}$ και $\frac{\partial F}{\partial y}$). Το σύστημα αυτό λύνεται με αντικατάσταση ή με τη μέθοδο του Cramer: η ορίζουσα του συστήματος είναι η

$$D = \begin{vmatrix} 1 & v \\ 1 & u \end{vmatrix} = u - v$$

και αρα το σύστημα έχει μοναδική λύση αν και μόνο αν $u \neq v$. Αυτή είναι και η συνθήκη για να είναι αντιστρέψιμος ο μετασχηματισμός $(u, v) \to (x(u, v), y(u, v))$. Άρα, για $u \neq v$, έχουμε

$$\frac{\partial F}{\partial x} = \frac{\begin{vmatrix} \frac{\partial h}{\partial u} & v \\ \frac{\partial h}{\partial v} & u \end{vmatrix}}{D} = \frac{1}{u-v}\left(u\frac{\partial h}{\partial u} - v\frac{\partial h}{\partial v}\right)$$

και

$$\frac{\partial F}{\partial y} = \frac{\begin{vmatrix} 1 & \frac{\partial h}{\partial u} \\ 1 & \frac{\partial h}{\partial v} \end{vmatrix}}{D} = \frac{1}{u-v}\left(\frac{\partial h}{\partial v} - \frac{\partial h}{\partial u}\right).$$

5 Έστω η παραγωγίσιμη συνάρτηση $f : \mathbb{R} \to \mathbb{R}$. Τότε η $u = f\left(\dfrac{x+y}{x-y}\right)$ ικανοποιεί την

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

Απάντηση

Αν $z(x, y) = \dfrac{x+y}{x-y}$, τότε

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{-2y}{(x-y)^2} \quad \text{και} \quad \frac{\partial u}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{2x}{(x-y)^2}.$$

Συνεπώς,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

6 Να υπολογίσετε την $\left.\dfrac{\partial F}{\partial t}\right|_{t=0}$, για $F(x, y) = x^2y - y^2$ και $x = x(t) = \sin t,\ y = y(t) = e^t$.

Απάντηση

$$F'(t) = x'(t)\frac{\partial F}{\partial x} + y'(t)\frac{\partial F}{\partial y} = \cos t\cdot(2x(t)y(t)) + e^t\cdot(x^2(t) - 2y(t))$$

και αρα

$$F'(0) = \cos 0(2\sin(0)e^0) + e^0(\sin^2(0) - 2e^0) = -2.$$