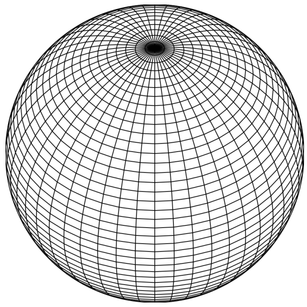
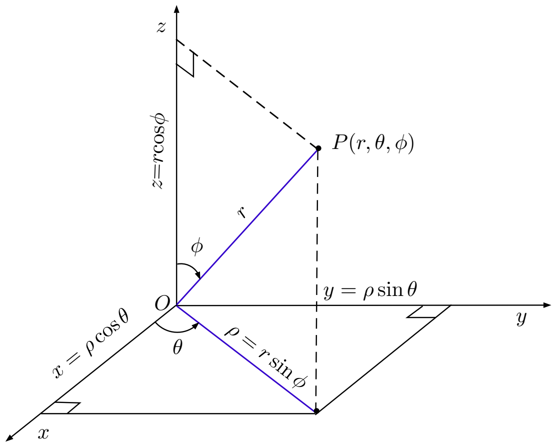


Σφαιρικές συντεταγμένες (στον $\mathbb{R}^3$)



Από το πιο πάνω σχήμα έπεται ότι

$$\cos\phi = \frac{z}{r} \Longrightarrow z = r\cos\phi, \qquad \sin\phi = \frac{\rho}{r} \Longrightarrow r\sin\phi = \rho$$

και αρα

$$\cos\theta = \frac{x}{\rho} = \frac{x}{r\sin\phi} \Longrightarrow x = r\sin\phi\cos\theta \quad \text{και} \quad \sin\theta = \frac{y}{\rho} = \frac{y}{r\sin\phi} \Longrightarrow y = r\sin\phi\sin\theta.$$

Έτσι,

$$\begin{cases} x = r\sin\phi\cos\theta, & 0 \le \phi \le \pi \\ y = r\sin\phi\sin\theta, & 0 \le \theta \le 2\pi \\ z = r\cos\phi & 0 \le r < +\infty \end{cases}$$

δηλ. έχουμε ενα μετασχηματισμό $f : [0, +\infty) \times [0, 2\pi] \times [0, \pi] \to \mathbb{R}^3$ με

$$f(r, \theta, \phi) = (x(r, \theta, \phi), y(r, \theta, \phi), z(r, \theta, \phi)) := (r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)$$

Θα υπολογίσουμε την ορίζουσα του Ιακωβιανού πίνακα J του πιο πάνω μετασχηματισμού:

$$\begin{aligned}
\det(J) &= \det\left(\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\right) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ \sin\phi\sin\theta & r\sin\phi\cos\theta & r\cos\phi\sin\theta \\ \cos\phi & 0 & -r\sin\phi \end{vmatrix} \\
&= r^2 \begin{vmatrix} \sin\phi\cos\theta & -\sin\phi\sin\theta & \cos\phi\cos\theta \\ \sin\phi\sin\theta & \sin\phi\cos\theta & \cos\phi\sin\theta \\ \cos\phi & 0 & -\sin\phi \end{vmatrix} \\
&= r^2[\sin\phi(-\sin^2\phi\cos\theta) - \sin\phi\sin\theta(\sin^2\phi\sin\theta) + \cos\phi(-\sin^2\theta\cos\phi\sin\phi - \sin\phi\cos\phi\cos^2\theta)] \\
&= r^2[-\sin^3\phi\cos^2\theta - \sin^3\phi\sin^2\theta - \cos^2\phi\sin^2\theta\sin\phi - \sin\phi\cos^2\phi\cos^2\theta] \\
&= r^2[-\sin^3\phi - \cos^2\phi\sin\phi(\sin^2\theta + \cos^2\theta)] \\
&= -r^2\sin\phi \cdot (\sin^2\phi + \cos^2\phi) = -r^2\sin\phi
\end{aligned}$$