Solutions of the exercises from

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Edward B. Saff, Arthur David Snider

Fundamentals of Complex Analysis
with Applications to
Engineering, Science, and Mathematics

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Chapter 4-Complex Integration

Exercises 4.5-Cauchy's Integral Formulas and Its Consequences
Theory:

Theorem 0.1. Cauchy-Goursat

Let $\Omega \subseteq \mathbb{C}$ simply connected open domain. Let f analytic in Ω. If γ is a piecewise smooth (C^1) loop in Ω, then

$$\int_\gamma f(z)\, dz = 0.$$

Theorem 0.2. Cauchy Integral Formula

Let f be analytic on a simply connected domain Ω. Suppose that $z_0 \in \Omega$ and γ is a simple loop (closed curve) oriented in the counterclockwise direction in Ω that encloses z_0. Then

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{\zeta - z_0} d\zeta$$

Theorem 0.3. Generalized Cauchy Integral Formula

Let f be analytic on a simply connected domain Ω. Then f has derivatives of all orders in Ω (which are analytic in Ω) and for any $z_0 \in \Omega$ we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

for any $n \in \mathbb{N}$, where γ is a simple loop (closed curve) oriented in the counterclockwise direction in Ω that encloses z_0.

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Problem 1

[Answer] Since f is analytic inside and on the simple closed contour Γ, (assumed positively oriented) there exists a simply connected set D such that $[\Gamma] \subset D$, and that $z_0 \notin D$. Then the function

$$F : D \to \mathbb{C} \qquad \text{with} \qquad \zeta \mapsto F(\zeta) = \frac{f(\zeta)}{\zeta - z_0}$$

is analytic and since D is simply connected, Cauchy's Theorem yields

$$\frac{1}{2\pi i} \int_\Gamma F(\zeta)\, d\zeta = 0.$$

Problem 2

[Answer] Γ is assumed positively oriented and let D be the portion of the plane contained inside Γ. By Cauchy's integral formula for the analytic function $h := f - g$, we have that

$$h(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{h(\zeta)}{\zeta - z}\, d\zeta, \quad \forall z \in D$$

i.e.

$$h(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\zeta) - g(\zeta)}{\zeta - z}\, d\zeta, \quad \forall z \in D$$

i.e. (since $f = g$ on Γ) $h(z) = 0 \quad \forall z \in D$. Thus

$$f(z) = g(z) \quad \forall z \in D$$

Problem 3

[Answer] (a) The function $f(z) = \sin(3z)$, $z \in \mathbb{C}$ is analytic (entire) and $z = \frac{\pi}{2} \in \text{int}(\Delta(0,2)) = C(0,2)$. Then, Cauchy's integral formula yields

$$f\left(\frac{\pi}{2}\right) = \frac{1}{2\pi i} \oint_{C+} \frac{f(\zeta)}{\zeta - \frac{\pi}{2}}\, d\zeta = \frac{1}{2\pi i} \oint_{C+} \frac{\sin(3\zeta)}{\zeta - \frac{\pi}{2}}\, d\zeta$$

and thus

$$\oint_{C+} \frac{\sin(3\zeta)}{\zeta - \frac{\pi}{2}}\, d\zeta = 2\pi i \cdot f\left(\frac{\pi}{2}\right) = 2\pi i \cdot \sin\left(\frac{3\pi}{2}\right) = -2\pi i \cdot \sin\left(\frac{\pi}{2}\right) = -2\pi i.$$

(b) Write

$$\oint_C \frac{ze^z}{2z - 3}\, dz = \frac{1}{2} \oint_C \frac{ze^z}{z - \frac{3}{2}}\, dz$$

The function $f(z) = ze^z$, $z \in \mathbb{C}$ is analytic (entire) and $z = \frac{3}{2} \in \text{int}\Delta(0.2) = C(0,2)$. Then, Cauchy's integral formula yields

$$f\left(\frac{3}{2}\right) = \frac{1}{2\pi i} \oint_{C+} \frac{f(\zeta)}{\zeta - \frac{3}{2}}\, d\zeta = \frac{1}{2\pi i} \oint_{C+} \frac{\zeta e^\zeta}{\zeta - \frac{3}{2}}\, d\zeta$$

and thus

$$\oint_{C^+} \frac{\zeta e^{\zeta}}{2\zeta - 3}\, d\zeta = \frac{1}{2} \cdot 2\pi i \cdot f\left(\frac{3}{2}\right) = \frac{3\pi e^{\frac{3}{2}}}{2}$$

(c) First note that we can decompose $1/(z^3 + 9z)$:

$$\frac{1}{z^3 + 9z} = \frac{1}{z(z^2+9)} = \frac{1}{z(z+3i)(z-3i)} = \frac{1}{9} \cdot \left[\frac{1}{z} + \frac{1}{2(z+3i)} + \frac{1}{2(z-3i)}\right]$$

and thus,

$$\oint_{C^+} \frac{\cos z}{z^3 + 9z}\, dz = \frac{1}{9} \oint_{C^+} \frac{\cos z}{z} dz + \frac{1}{18} \oint_{C^+} \frac{\cos z}{z+3i} dz + \frac{1}{18} \oint_{C^+} \frac{\cos z}{z-3i} dz$$

Consider the function $f(z) = \cos z$, $z \in \mathbb{C}$ which is entire. Then, since the points $\pm 3i$ do not lie inside the interior of C, we have that

$$\oint_{C^+} \frac{\cos z}{z+3i} dz = 0 = \oint_{C^+} \frac{\cos z}{z-3i} dz$$

and thus

$$\oint_{C^+} \frac{\cos z}{z^3+9z}\, dz = \frac{1}{9} \cdot \oint_{C^+} \frac{f(\zeta)}{\zeta - 0} d\zeta = \frac{1}{9} \cdot 2\pi i \cdot f(0) = \frac{2\pi i}{9} \cos 0 = \frac{2\pi i}{9}$$

(d) Define $f(z) = 5z^2 + 2z + 1$, $z \in \mathbb{C}$. Then

$$\oint_{C^+} \frac{5\zeta^2 + 2\zeta + 1}{(\zeta - i)^3}\, d\zeta = \oint_{C^+} \frac{f(\zeta)}{(\zeta - i)^3}\, d\zeta$$

By the Generalized Cauchy integral formula (since $i \in \text{int}(\Delta(0,2)) = C(0,2)$, we get:

$$f^{(2)}(i) = \frac{2!}{2\pi i} \oint_{C^+} \frac{f(\zeta)}{(\zeta - i)^{2+1}}\, d\zeta,$$

i.e.

$$\oint_{C^+} \frac{f(\zeta)}{(\zeta - i)^3}\, d\zeta = \pi i \cdot f^{(2)}(i) = 10\pi i$$

for since $f^{(2)}(z) = 10$, $\forall z \in \mathbb{C}$.

(e) Define $f(z) = e^{-z}$, $z \in \mathbb{C}$ which is entire. Then

$$\oint_{C^+} \frac{e^{-\zeta}}{(\zeta+1)^2}\, d\zeta = \oint_{C^+} \frac{f(\zeta)}{(\zeta - (-1))^{1+1}}\, d\zeta$$

By the Generalized Cauchy integral formula (since $(-1) \in \text{int}(\Delta(0,2)) = C(0,2)$, we get:

$$f^{(1)}(-1) = \frac{1!}{2\pi i} \oint_{C^+} \frac{f(\zeta)}{(\zeta+1)^{1+1}}\, d\zeta,$$

i.e.

$$\oint_{C^+} \frac{f(\zeta)}{(\zeta+1)^{1+1}}\, d\zeta = 2\pi i \cdot f^{(1)}(-1) = -2\pi e,$$

for since $f^{(1)}(z) = -e^{-z}$, $\forall z \in \mathbb{C}$.

(f) First note that we can decompose $1/(z^2(z+4))$:

$$\frac{1}{z^2(z+4)} = \frac{1}{16}\left[\frac{1}{z-4} - \frac{1}{z} - \frac{4}{z^2}\right]$$

and thus,

$$\oint_{C^+} \frac{\sin z}{z^2(z+4)}\,dz = \frac{1}{16}\oint_{C^+}\left[\frac{\sin z}{z-4} - \frac{\sin z}{z} - \frac{4\sin z}{z^2}\right]dz$$

Define $f(z) = \sin z$, $z \in \mathbb{C}$. The point $z = 4$ does not lie in the interior of C and thus $\oint_{C^+}\frac{f(\zeta)}{z-4}d\zeta = 0$ and Cauchy's (generalized) Integral formula yields

$$\oint_{C^+}\frac{\sin\zeta}{\zeta^2}d\zeta = \oint_{C^+}\frac{\sin\zeta}{(\zeta-0)^2}d\zeta = 2\pi i\cdot f^{(1)}(0) = 2\pi i\cdot\frac{d\sin z}{dz}\Big|_{z=0} = 2\pi i$$

and

$$\oint_{C^+}\frac{\sin\zeta}{\zeta}d\zeta = \oint_{C^+}\frac{\sin\zeta}{\zeta-0}d\zeta = 2\pi i\cdot f(0) = 2\pi i\cdot\sin 0 = 0.$$

Putting it all together,

$$\oint_{C^+}\frac{\sin z}{z^2(z-4)}\,dz = -\frac{\pi i}{2}$$

Problem 4

[Answer] First note that we can decompose $1/(z^3+2z^2) =$:

$$\frac{1}{z^3+2z^2} = \frac{1}{z^2(z+2)} = \frac{1}{4}\left[\frac{1}{z+2} - \frac{1}{z} + \frac{2}{z^2}\right]$$

Consider the function $f(z) = z + i$, $z \in \mathbb{C}$. f is an entire function. We have

$$\oint_C \frac{z+i}{z^3+2z^2}dz = \frac{1}{4}\oint_C\frac{f(z)}{z-(-2)}dz - \frac{1}{4}\oint_C\frac{f(z)}{z-0}dz + \frac{1}{2}\oint_C\frac{f(z)}{(z-0)^2}dz$$

(a) Let $C = \{|z| = 1\} = C(0,1)$ traversed once counterclockwise. Then since $z = -2$ lies outside the interior of C, we have $\oint_C\frac{f(z)}{z+2}dz = 0$. Also, Cauchy's (generalized) Integral formula yields

$$\oint_C\frac{f(z)}{(z-0)^2}dz = \frac{2\pi i}{1!}\cdot f'(0) = 2\pi i$$

and

$$\oint_C\frac{f(z)}{z-0}dz = 2\pi i f(0) = -2\pi$$

Putting it all together,

$$\oint_C\frac{f(z)}{z^3+2z^2}dz = \frac{\pi}{2} + \pi\cdot i.$$

(b) Let $C = \{|z+2-i| = 2\} = \{|z-(i-2)| = 2\}$ traversed once counterclockwise. Then we have

$$\oint_C\frac{z+i}{z^2}dz = \oint_C\frac{\frac{z+i}{z}}{z-0}dz = 0, \qquad \oint_C\frac{z+i}{z}dz = 0$$

and

$$\oint_C \frac{z+i}{z-(-2)}dz \overset{Cauchy}{=} 2\pi i \cdot f(-2) = 2\pi i \cdot (i-2)$$

Putting it all together,

$$\oint_C \frac{f(z)}{z^3+2z^2}dz = -\pi \frac{2i+1}{2}$$

(c) Let $C = \{|z-2i| = 1\}$ traversed once counterclockwise. Then since the points $z = 0, -2$ are outside the closure of C, the function $z \mapsto \frac{z+i}{z^3+2z^2}$ is analytic in the interior of C, thus it has a primitive there and so

$$\oint_C \frac{f(z)}{z^3+2z^2}dz = 0$$

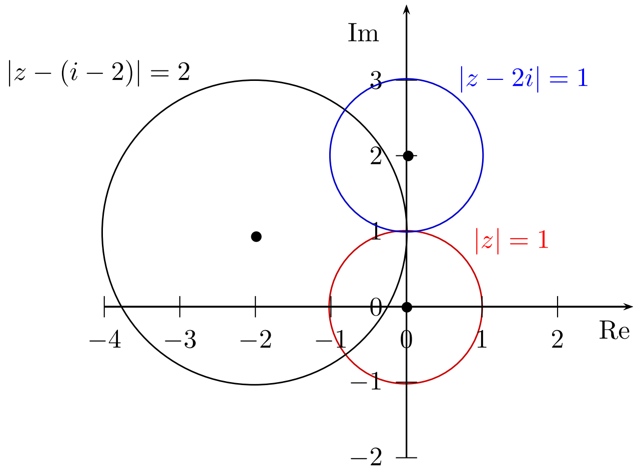

Figure 1: Exercise 4.5/4

Problem 5

[Answer]

$$C : \frac{x^2}{4} + \frac{y^2}{9} = 1 \iff C : \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

Consider a slightly larger simply connected set D containing C in its interior, for example

$$D = \left\{(x, y) \in \mathbb{C} \,:\, \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 < \frac{3}{2}\right\}.$$

Then, the function $f(\zeta) = \zeta^2 - \zeta + 2$ is analytic in D and thus Cauchy's integral formula yields:

$$f(z) = \frac{1}{2\pi i} \oint_{C+} \frac{\zeta^2 - \zeta + 2}{\zeta - z} d\zeta \quad (z \in \text{int}(C))$$

i.e. $G(z) = 2\pi i f(z)$, $\forall z \in \text{int}(C)$ and thus

$$G(z) = 2\pi \cdot i(z^2 - z + 2), \forall z \in \text{int}(C).$$

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Derivation of this last equation gives

$$G'(z) = 2\pi \cdot i(2z-1),\ \forall z \in \text{int}(C).$$

and

$$G''(z) = 4\pi \cdot i,\ \forall z \in \text{int}(C).$$

Thus, $G(1) = 4\pi i$, $G'(i) = 2\pi \cdot i(2i-1)$ and $G''(-i) = 4\pi \cdot i$.

Problem 6

[Answer] Since

$$\frac{e^{iz}}{(z^2+1)^2} = \frac{e^{iz}}{(z-i)^2(z+i)^2},$$

the function $f : \mathbb{C} - \{\pm i\} \to \mathbb{C}$ with $f(z) = \frac{e^{iz}}{(z^2+1)^2}$ is analytic. Now consider two circles C_1 and C_2 around the points $z = i$ and $z = -i$ respectively, traversed once counterclockwise. Denote (see figure (2)) γ_i, $i = 1, 2, \dots, 9$. Then, if

$$\Gamma_1^+ = \Gamma_1^+ \oplus \Gamma_2^+ \oplus \dots \oplus \Gamma_6^+$$

and

$$\Gamma_2^- = \Gamma_9^- \oplus \Gamma_6^- \oplus \Gamma_7^- \oplus \Gamma_4^- \oplus \Gamma_8^- \oplus \Gamma_2^-$$

by cancelling opposite terms, we get

$$\oint_\Gamma f(z)\,dz = \oint_{\Gamma_1^+} f(z)\,dz + \oint_{\Gamma_2^-} f(z)\,dz - \oint_{C_1^+} f(z)\,dz - \oint_{C_2^+} f(z)\,dz$$

If U_1 and U_2 is the interior of the contours Γ_1^+ and Γ_2^- respectively, let D_1 and D_2 be slightly larger than U_1 and U_2 respectively (such that the do not contain and of the points $z = \pm i$). Then they are simply connected domains and $\Gamma_1^+ \subset D_1$ and $\Gamma_2^- \subset D_1$. Cauchy's integral formula yields then

$$\oint_{\Gamma_1^+} f(z)\,dz = 0 = \oint_{\Gamma_2^-} f(z)\,dz$$

Thus

$$\oint_\Gamma f(z)\,dz = -\left(\oint_{C_1^+} f(z)\,dz + \oint_{C_2^+} f(z)\,dz\right) = \oint_{C_1^-} f(z)\,dz + \oint_{C_2^-} f(z)\,dz$$

Now,

$$\begin{aligned}\oint_{C_1^-} f(z)\,dz &= \oint_{C_1^-} \frac{\frac{e^{iz}}{(z+i)^2}}{(z-i)^2}\,dz \overset{Cauchy}{=} \frac{2\pi i}{1!} \cdot \frac{d}{dz}\left(\frac{e^{iz}}{(z+i)^2}\right)\Big|_{z=i} \\ &= 2\pi i \cdot \left(\frac{e^{iz}(iz-3)}{(z+i)^3}\right)\Big|_{z=i} = \frac{\pi}{e}\end{aligned}$$

and (similarly)

$$\begin{aligned}\oint_{C_2^-} f(z)\,dz &= \oint_{C_2^-} \frac{\frac{e^{iz}}{(z-i)^2}}{(z+i)^2}\,dz \overset{Cauchy}{=} \frac{2\pi i}{1!} \cdot \frac{d}{dz}\left(\frac{e^{iz}}{(z-i)^2}\right)\Big|_{z=-i} \\ &= 2\pi i \cdot \left(\frac{e^{iz}(iz-1)}{(z-i)^3}\right)\Big|_{z=-i} = 2\pi i \cdot 0 = 0\end{aligned}$$

Finally,

$$\oint_{\Gamma} f(z)\,dz = \oint_{C_1^-} f(z)\,dz = \frac{\pi}{e}$$

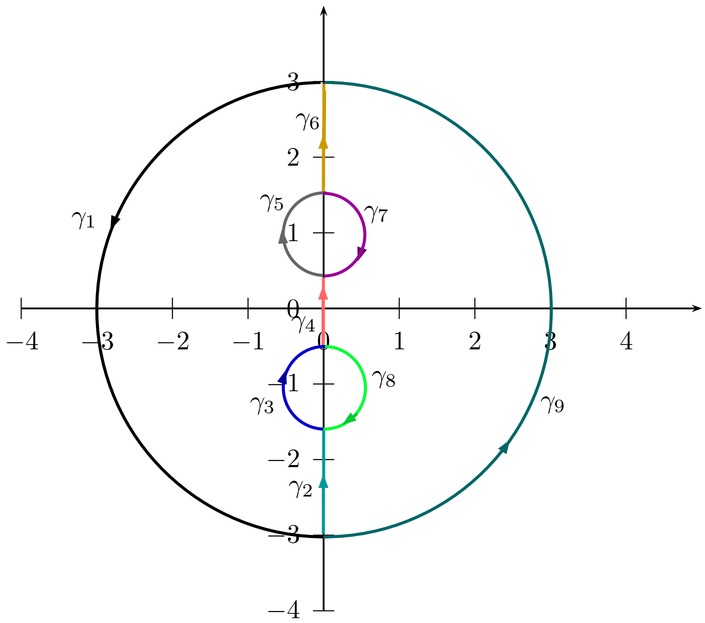

Figure 2: Exercise 6

Problem 8

[Answer] Since f is analytic in analytic inside and on the circle $|z - z_0| = r$, we can consider a slightly larger simply connected open set U containing the (closed) disc $\Delta(z_0, r) = \{|z - z_0| \leq r\}$ in its interior. Then, Cauchy's integral formula yields

$$f(z_0) = \frac{1}{2\pi i} \int_{C(z_0,r)} \frac{f(\zeta)}{\zeta - z_0}\,d\zeta.$$

If we parameterize $C(z_0, r)$ using polar coordinates:

$$\zeta(\theta) - z_0 = re^{i\theta}, \quad \theta \in [0, 2\pi),$$

then

$$\frac{1}{2\pi i} \int_{C(z_0,r)} \frac{f(\zeta)}{\zeta - z_0}\,d\zeta = \frac{1}{2\pi i} \int_{C(z_0,r)} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta}\,d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta$$

and thus the first formula is proved.
For the second formula, under the above setting, Cauchy's generalized integral formula yields (for

$n = 1, 2, 3, \ldots$):

$$\begin{aligned} f^{(n)}(z_0) &= \frac{n!}{2\pi i}\int_{C(z_0,r)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta \quad = \quad \frac{n!}{2\pi i}\int_{C(z_0,r)} \frac{f(z_0 + re^{i\theta})}{(re^{i\theta})^{n+1}} ire^{i\theta}\, d\theta \\ &= \frac{n!}{2\pi r^n}\int_0^{2\pi} f(z_0 + re^{i\theta})e^{-in\theta}\, d\theta \end{aligned}$$

Problem 9

[Answer] By hypothesis, f is bounded on the unit circle. Then since $z = 0$ lies in the interior of the disc $\Delta(0, 1)$, Cauchy's integral formula yields

$$\begin{aligned} f(0) &= \frac{1}{2\pi i}\int_{C(0,1)} \frac{f(\zeta)}{\zeta}\, d\zeta \\ \Longrightarrow |f(0)| &= \frac{1}{2\pi}\left|\int_{C(0,1)} \frac{f(\zeta)}{\zeta}\, d\zeta\right| \leq \frac{1}{2\pi} \cdot \sup_{|\zeta|=1} \frac{|f(\zeta)|}{|\zeta|} \cdot \text{length}(C(0,1)) \leq M \cdot \frac{2\pi}{2\pi} = M \end{aligned}$$

Now, Cauchy's generalized integral formula yields (for $n = 1, 2, 3, \ldots$):

$$\begin{aligned} f^{(n)}(0) &= \frac{n!}{2\pi i}\int_{C(0,1)} \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta \\ \Longrightarrow |f^{(n)}(0)| &= \frac{n!}{2\pi}\left|\int_{C(0,1)} \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta\right| \leq \frac{n!}{2\pi}\int_{C(0,1)} \frac{|f(\zeta)|}{|\zeta|^{n+1}}\, d\zeta \\ &\leq \frac{n!}{2\pi} \cdot \sup_{|\zeta|=1} |f(\zeta)| \cdot \text{length}(C(0,1)) \leq M \cdot \frac{2\pi}{2\pi} \cdot n! = M \cdot n! \end{aligned}$$

For $n = 1$, the above gives $|f'(0)| \leq M$.

Problem 10

[Answer] Let z_0 not on Γ. We may assume the simple closed contour Γ lies completely inside a simply connected open domain U and such that f is analytic in U. Also, by Cauchy's Theorem, f' is also analytic in U.

Case 1: If z_0 belongs outside of U, then both $z \mapsto \frac{f(z)}{(z-z_0)^2}$ and $z \mapsto \frac{f'(z)}{z-z_0}$ are analytic in U. Thus, by Cauchy's Theorem,

$$\int_\Gamma \frac{f'(z)}{z - z_0} dz = 0 = \int_\Gamma \frac{f(z)}{(z - z_0)^2} dz$$

Case 2: If z_0 belongs in the interior of Γ, then by Cauchy's Theorem applied on f',

$$2\pi i \cdot f'(z_0) = \int_\Gamma \frac{f'(z)}{z - z_0} dz \tag{1}$$

and by the Generalized Cauchy formula,

$$1! \cdot 2\pi i \cdot f'(z_0) = \int_\Gamma \frac{f(z)}{(z - z_0)^2} dz \tag{2}$$

Thus, (1) and (2) yield

$$\int_\Gamma \frac{f'(z)}{z-z_0}dz = \int_\Gamma \frac{f(z)}{(z-z_0)^2}dz.$$

Problem 11

[Answer] Since f is analytic in D, we know that its derivatives of all orders (with respect to z) exist. Specifically, for $z_0 = (x_0, y_0) \in D$, we have

$$f'(z_0) = f'(x_0, y_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)$$

and thus

$$f''(z_0) = f''(x_0, y_0) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)(x_0, y_0) + i\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)(x_0, y_0) = \frac{\partial^2 u}{\partial x^2}(x_0, y_0) + i\frac{\partial^2 v}{\partial x^2}(x_0, y_0)$$

Then $\partial^2 u/\partial x^2$ is the real part of the analytic function f'', thus harmonic (on D).
Note: See Theorem 17 from the book

Problem 12

[Answer] We will go along the lines as with the proof with Theorem 15 in the book.
Let $z \in \text{int}(\Gamma)$. We will show that $\lim_{h\to 0} J(h; z) = 0$, where

$$J(h; z) = \frac{H(z+h) - H(z)}{h} - 2\int_\Gamma \frac{g(\zeta)}{(\zeta - z)^3}d\zeta$$

We first write $J(h; z)$ as one integral and then use the ML property for integrals so as to bound it:

$$\begin{aligned}
J(h; z) &= \frac{H(z+h) - H(z)}{h} - 2\int_\Gamma \frac{g(\zeta)}{(\zeta - z)^3}d\zeta \\
&= \frac{1}{h}\left[H(z+h) - H(z)\right] - 2\int_\Gamma \frac{g(\zeta)}{(\zeta - z)^3}d\zeta \\
&= \frac{1}{h}\left[\int_\Gamma \frac{g(\zeta)}{(\zeta - (z+h))^2}d\zeta - \int_\Gamma \frac{g(\zeta)}{(\zeta - z)^2}d\zeta\right] - 2\int_\Gamma \frac{g(\zeta)}{(\zeta - z)^3}d\zeta \\
&= \frac{1}{h}\int_\Gamma g(\zeta)\left[\frac{1}{(\zeta - z - h)^2} - \frac{1}{(\zeta - z)^2} - \frac{2h}{(\zeta - z)^3}\right]d\zeta \\
&= \frac{1}{h}\int_\Gamma \frac{g(\zeta)[(\zeta - z)^3 - (\zeta - z - h)^2(\zeta - z) - 2h(\zeta - z - h)^2]}{(\zeta - z - h)^2(\zeta - z)^3}d\zeta \\
\text{calculations} \quad &= \frac{1}{h}\int_\Gamma \frac{g(\zeta)\cdot h^2 \cdot (3\zeta - 3z - 2h)}{(\zeta - z - h)^2(\zeta - z)^3}d\zeta \\
&= \int_\Gamma h \cdot \frac{3(\zeta - z) - 2h}{(\zeta - z - h)^2(\zeta - z)^3}\cdot g(\zeta)d\zeta
\end{aligned}$$

Now let

$$d := \text{dist}(z, \Gamma) = \inf_{w \in [\Gamma]} \{|w - z|\} \qquad \text{and} \qquad r := \sup_{w \in [\Gamma]} \{|w - z|\}.$$

We can choose w.l.o.g. h such that $0 < |h| < \frac{d}{2}$.Then, $(z + h) \in [\Gamma]$ and if $\zeta \in [\Gamma]$, then

$$|\zeta - z - h| \geq |\zeta - z| - |h| > d - \frac{d}{2} = \frac{d}{2}$$

and thus

$$\frac{1}{|\zeta - z - h|^2} \leq \frac{4}{d^2}.$$

Then for any $\zeta \in [\Gamma]$,

$$\left| h \cdot \frac{3(\zeta - z) - 2h}{(\zeta - z - h)^2(\zeta - z)^3} \cdot g(\zeta) \right| \leq M \cdot \frac{d}{2} \cdot \frac{3|\zeta - z| + 2|h|}{|\zeta - z - h|^2 \cdot |\zeta - z|^3} < \frac{2M(3r + d)}{d^4}$$

Hence,

$$|J(h; z)| \leq \frac{2M(3r + d)}{d^4} \cdot \text{Length}(\Gamma)$$

and since the right side of the above inequality tends to zero as we let $h \to 0^+$, the result follows.

Problem 13

[Answer] The function $g(z) = 1/z$ is analytic on $\mathbb{C} - \{0\}$ (the punctured plane) which is not simply connected.
Let $z \in \mathbb{C} - \{0\}$ and such that $z \notin [\Gamma]$. Then,

$$\begin{aligned}
\frac{1}{2\pi i} \int_{|\zeta|=1} \frac{g(\zeta)}{\zeta - z} d\zeta &= \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\frac{1}{\zeta}}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{d\zeta}{\zeta(\zeta - z)} \\
&= \frac{1}{2\pi i \cdot z} \int_{|\zeta|=1} \left[\frac{1}{\zeta - z} - \frac{1}{\zeta} \right] d\zeta \\
&= \frac{1}{2\pi i \cdot z} \int_{|\zeta|=1} \frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i \cdot z} \int_{|\zeta|=1} \frac{d\zeta}{\zeta} \\
&= \frac{1}{2\pi i \cdot z} \int_{|\zeta|=1} \frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i \cdot z} \cdot 2\pi i \\
&= \frac{1}{z} \cdot \left(\frac{1}{2\pi i} \int_{|\zeta|=1} \frac{d\zeta}{\zeta - z} \right) - \frac{1}{z}
\end{aligned}$$

But

$$\frac{1}{2\pi i} \int_{|\zeta|=1} \frac{d\zeta}{\zeta - z} = \begin{cases} 1, & z \in \text{int}([\Gamma]) \\ 0, & z \notin \text{int}([\Gamma]) \end{cases} = \begin{cases} 1, & |z| < 1 \\ 0 & |z| > 1 \end{cases}$$

and thus

$$\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{g(\zeta)}{\zeta - z}d\zeta = \begin{cases} 0, & |z|<1 \\ \\ -\frac{1}{z}, & |z|>1 \end{cases}$$

If $z = 0$, then

$$G(0) = \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{1}{\zeta^2}d\zeta = 0,$$

since the function $\zeta \mapsto 1/\zeta^2$ has a primitive on $\Gamma = \{|z| = 1\}$. Hence, if $w \in \mathbb{C}$ such that $w \in [\Gamma]$,

$$\lim_{\substack{z\to w\\|z|<1}} G(z) = \lim_{\substack{z\to w\\|z|<1}} \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{g(\zeta)}{\zeta - z}d\zeta = 0 \neq \frac{1}{w} = g(w)$$

and

$$\lim_{\substack{z\to w\\|z|<1}} G(z) = \lim_{\substack{z\to w\\|z|<1}} \left(-\frac{1}{z}\right) = -\frac{1}{w} \neq \frac{1}{w} = g(w)$$

This does not violates Cauchy's Theorem, for since the function $g(z) = 1/z$ is analytic on $\mathbb{C} - \{0\}$ (the punctured plane) which is not simply connected and thus the Theorem cannot be applied.

Problem 14

[Answer] If $z \in \text{int}([\Gamma])$, then by Cauchy's Theorem (since $f(z) = \cos z$ is entire), $G(z) = f(z) = \cos z$.
If $z \notin \overline{\text{int}([\Gamma])}$, we can consider w.l.o.g. a simply connected domain U such that $\text{dist}([\Gamma], U) > 0$. Then, the function $\zeta \mapsto \frac{\cos\zeta}{\zeta - z}$, $\zeta \in U$ is analytic and thus possesses a primitive function there, hence $G(z) = 0$. Then:

(1) $$\lim_{\substack{z\to 2+3i\\z\in\text{int}([\Gamma])}} G(z) = \lim_{\substack{z\to 2+3i\\z\in\overline{\text{int}([\Gamma])}}} \cos z = \cos(2+3i)$$

(2) $$\lim_{\substack{z\to 2+3i\\z\notin\overline{\text{int}([\Gamma])}}} G(z) = \lim_{\substack{z\to 2+3i\\z\notin\text{int}([\Gamma])}} 0 = 0$$

Problem 15

[Answer] Since f is analytic at each point of the closed disc $|z| \leq 1$ we have that the function F is analytic on $\{0 < |z| \leq 1\}$. We will show that F is analytic also at $z = 0$. Define the function $g(\zeta) = \frac{f(\zeta)}{\zeta}$, $\zeta \in \{|\zeta| = 1\}$. This function is continuous, thus by Theorem 15, the function

$$G(z) := \frac{1}{2\pi i}\oint_{|\zeta|=1}\frac{f(\zeta)/\zeta}{\zeta - z}d\zeta \quad (|z|<1)$$

is analytic. In particular, it is analytic at the origin. Now, for $z \in \{0 < |z| < 1\}$ we have

$$\begin{aligned} G(z) &= \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{f(\zeta)}{\zeta(\zeta - z)} d\zeta \\ &= \frac{1}{2\pi i \cdot z} \oint_{|\zeta|=1} f(\zeta) \left[\frac{1}{\zeta - z} - \frac{1}{\zeta} \right] d\zeta \\ &= \frac{1}{z} \cdot \left[\frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{f(\zeta)}{\zeta} d\zeta \right] \\ &\overset{\text{Cauchy}}{=} \frac{1}{z} \cdot (f(z) - \overbrace{f(0)}^{=0}) = \frac{f(z)}{z} = F(z) \end{aligned}$$

Now,

$$G(0) = \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{f(\zeta)}{\zeta^2} d\zeta = \frac{1!}{2\pi i} \oint_{|\zeta|=1} \frac{f(\zeta)}{(\zeta - 0)^2} d\zeta \overset{\text{Cauchy}}{=} f'(0) = F(0)$$

We've showed that $G = F$ on $\{|z| < 1\}$. Finally, since G is analytic at the origin, F is also analytic there.

Problem 16

[Answer]

(1) Since f is never zero (on D), i.e. $f(z) \neq 0, \forall z \in D$, the function $g(z) = \frac{f'(z)}{f(z)}$, $z \in D$ is well defined. Also, since f is analytic (in D), f' is also analytic, hence g is analytic (in D).

(2) Let γ be a simple closed contour inside D. Then, since D is simply connected and f'/f is analytic in D,

$$\int_\gamma \frac{f'(\zeta)}{f(\zeta)} d\zeta = 0.$$

[If one wants to be more precise, using the language of Algebraic topology: $\gamma \sim \{z_0\}$, where z_0 is a point on $[\gamma]$, i.e. γ is homotopic to one of the points of its trace. Hence, since D is simply connected, $\int_\gamma \frac{f'(\zeta)}{f(\zeta)} d\zeta = \int_{\{z_0\}} \frac{f'(\zeta)}{f(\zeta)} d\zeta = 0$].

We thus have proved that $\int_\gamma \frac{f'(\zeta)}{f(\zeta)} d\zeta = 0$ on every simple loop in D, hence by Cauchy's Theorem, the function $g = f'/f$ has an antiderivative in D, say H :

$$H'(z) = g(z) = \frac{f'(z)}{f(z)}, \quad \forall z \in D. \tag{3}$$

(3) For $z \in D$,

$$\begin{aligned} \left(f(z) \cdot e^{-H(z)} \right)' &= f'(z) \cdot e^{-H(z)} - f(z) \cdot H'(z) \cdot e^{-H(z)} \\ &\overset{(3)}{=} f'(z) \cdot e^{-H(z)} - f(z) \cdot \frac{f'(z)}{f(z)} \cdot e^{-H(z)} \\ &= f'(z) \cdot e^{-H(z)} - f'(z) \cdot e^{-H(z)} = 0 \end{aligned}$$

hence (since D is simply connected) $f \cdot e^{-H}$ is a constant function: $f \cdot e^{-H} = c$, i.e. $f = ce^{H}$ (in D).

(4) Since the exponential function is onto, there exists $\alpha \in \mathbb{C}$ such that $e^{\alpha} = c$, thus $f = e^{H+\alpha}$. In other words, the function $H + \alpha$ is a branch of $\log f$, analytic in D.

What this exercise says, is that by fixing a point $z_0 \in D$, then

$$f(z) = f(z_0) \cdot \exp\left(\int_{z_0}^{z} \frac{f'(\zeta)}{f(\zeta)} d\zeta \right), \ \forall z \in D.$$

i.e. the function

$$z \mapsto \log(f(z_0)) + \int_{z_0}^{z} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta$$

is an analytic branch of $\log f$ in D.

Problem 17

[Answer] Let $f(z) = z^3 - 1$, $z \in \Delta(0,1) = \{|z| < 1\}$. Then $f(z) \neq 0, \forall z \in \Delta(0,1)$ for since $z^3 = 1 \Longleftrightarrow |z| = 1$. Then, by Prob. 16, there exists an analytic function H such that $f = e^H$ in $\Delta(0,1)$, i.e. H is a branch of $\log f$, analytic in $\Delta(0,1)$. It is then straightforward that $H^{1/2}$ is a branch of $\log f^{1/2}$, analytic in $\Delta(0,1)$.
In other words, the (analytic) function $\widetilde{H} = e^{H/2}$ is an analytic square root, where H is an analytic branch of $\log f$ in $\Delta(0,1)$.

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