Chapter 1

Notes on Differential operators

We use the standard notation for multi-indices. Set

$$\mathbb{N}_0^n = \underbrace{\mathbb{N}_0 \times \mathbb{N}_0 \times \ldots \times \mathbb{N}_0}_{n-times},$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}_0^n$. Define

$$|\alpha| := \sum_{i=1}^{n} \alpha_i = \alpha_1 + \alpha_2 + \ldots + \alpha_n.$$

For any pair of multi indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n),\ \beta = (\beta_1, \beta_2, \ldots, \beta_n)$ define their sum as

$$\alpha + \beta := (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_n + \beta_n)$$

and each pair is ordered by the relationship

$$\alpha \leq \beta \overset{def.}{\Leftrightarrow} \alpha_i \leq \beta_i, \quad (1 \leq i \leq n).$$

For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, set

$$x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$$

and

$$\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_n^{\alpha_n},$$

where

$$\partial_i^{\alpha_i} = \frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}}, \quad (1 \leq i \leq n).$$

In other words,

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}.$$

For $X \subseteq \mathbb{R}^n$, then $C_c^\infty(X)$ or $\mathcal{D}(X)$ stands for the usual notation of the set of infinitely differentiable functions which have a compact support and $L^1_{loc}(X)$ for the set of locally (Lebesgue) integrable functions in X. If V a vector space on $\mathbb{C}$, then a linear functional(on V) ia a homeomorphism $u : V \to \mathbb{C}$. Linear forms (on V) become a vector space $\mathrm{Hom}(V, \mathbb{C})$ by acting on elements $\phi \in V$ as follows:

1. $\langle cu, \phi\rangle = c\langle u, \phi\rangle,\ \forall c \in \mathbb{C},\ u \in \mathrm{Hom}(V, \mathbb{C}),\ \phi \in V.$
2. $\langle u + u, \phi\rangle = \langle u, \phi\rangle + \langle v, \phi\rangle,\ \forall u, v \in \mathrm{Hom}(V, \mathbb{C}),\ \phi \in V.$

Distributions are linear functionals acting on the vector space $V = C_c^\infty(\mathbb{R}^n) \equiv \mathcal{D}(\mathbb{R})$.

Definition 1.0.1. *We define as Schwarz's space the set*

$$\mathcal{S}(\mathbb{R}^n) = \{\phi : \mathbb{R}^n \to \mathbb{C} \text{ or } \mathbb{R}\ :\ \forall \alpha, \beta \in \mathbb{N}_0^n,\ \sup_{x\in\mathbb{R}^n} |x^\beta \partial^\alpha \phi(x)| < \infty\}.$$

The elements of $\mathcal{S}(\mathbb{R}^n)$ share the property that $\lim_{|x|\to\infty} |x^\beta \partial^\alpha \phi(x)| = 0$ and thus we call this set as the set of rapidly decreasing functions. $\mathcal{S}(\mathbb{R}^n)$ is equipped with the following norm: for $k \in \mathbb{N}$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$, we set

$$\|\cdot\|_k : \mathcal{S}(\mathbb{R}^n) \to \mathbb{R}, \mathbb{C} \quad \textit{with} \quad \|\phi\|_k = \max_{|\alpha|+|\beta|=k} \sup_{x\in\mathbb{R}^n} |x^\beta \partial^\alpha \phi(x)|$$

In addition, we obtain the metric

$$d : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to \mathbb{R}, \mathbb{C} \quad \textit{with} \quad d(\phi, \psi) = \sum_{k=0}^{\infty} \frac{1}{2^k} \cdot \frac{\|\phi - \psi\|_k}{1 + \|\phi - \psi\|_k},$$

$(\phi, \psi \in \mathcal{S}(\mathbb{R}^n))$.

Definition 1.0.2. *Let $\phi \in \mathcal{S}(\mathbb{R}^n)$. For $\alpha, \beta \in \mathbb{Z}_+^n$ we set*

$$\|\phi\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} |x^\beta \partial^\alpha \phi(x)|.$$

We say that a sequence on $(\phi_n)_n \subset \mathcal{S}(\mathbb{R}^n)$ converges to the function ϕ on the space $\mathcal{S}(\mathbb{R}^n)$ if for $\alpha, \beta \in \mathbb{Z}_+^n$ we have $\|\phi_n - \phi\|_{\alpha,\beta} \xrightarrow{n} 0$, i.e.

$$\sup_{x\in\mathbb{R}^n} |x^\beta \partial^\alpha (\phi_n(x) - \phi(x))| \to 0.$$

A linear form $u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ is called continuous if there exist a constant $C \geq 0$ and $k, m \in \mathbb{Z}_+$ such that

$$|\langle u, \phi\rangle| \leq C \sum_{|\alpha|\leq m, |\beta|\leq k} \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta \phi| = C \sum_{|\alpha|\leq m, |\beta|\leq k} \|\phi\|_{\alpha,\beta}, \tag{1.1}$$

The linear continuous functionals on the space $\mathcal{S}(\mathbb{R}^n)$ are called tempered distributions. The set of all tempered distributions is denoted $(\mathcal{S}(\mathbb{R}^n))'$ and it is the dual space of $\mathcal{S}(\mathbb{R}^n)$. Schwarz's space shares some well known properties which we now state. Proofs of these results can be found on any book about distributions, such as [3].

(i) $C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ and $\overline{\mathcal{D}(\mathbb{R})} = \mathcal{S}(\mathbb{R}^n)$.

(ii) $\mathcal{S}(\mathbb{R}^n)$ is complete.

(iii) $\mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$.

(iv) A linear form u on $\mathcal{S}(\mathbb{R}^n)$ is continuous if and only if it is sequentially continuous: if $(\phi_n)_n \subset \mathcal{S}(\mathbb{R}^n)$ such that $\phi_n \to 0$ in $\mathcal{S}(\mathbb{R}^n)$, then $\langle u, \phi_n\rangle \to 0$.

1.0.1 The Fourier transform

Division of a distribution by a polynomial in dimension 2 gives rise to some divergent boundary integrals. In order to express the elementary solution derived from the division in terms of the dual space of tempered distributions, we use the well known Fourier transform. Thus, for the sake of completeness, we outline some of the basic properties for the Fourier transform and we use throughout this thesis.

Definition 1.0.3. *Let $f : \mathbb{R}^n \to \mathbb{C}$ be a function. We call as the Fourier transform of f the function $\widehat{f} \equiv \mathcal{F}(f) : \mathbb{R}^n \to \mathbb{C}$ with*

$$\lambda \mapsto \widehat{f}(\lambda) := \int_{\mathbb{R}^n} e^{-2\pi i\lambda\cdot x} f(x)\, dx, \tag{1.2}$$

where $\lambda \cdot x = \langle \lambda, x\rangle = \sum_{i=1}^{n} \lambda_i x_i$, if $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $x = (x_1, \ldots, x_n)$. The inverse Fourier transform of f is the function $\widehat{G} \equiv \mathcal{G}(f) : \mathbb{R}^n \to \mathbb{C}$ with

$$x \mapsto \widehat{G}(x) := \int_{\mathbb{R}^n} e^{2\pi i\lambda\cdot x} f(\lambda)\, d\lambda. \tag{1.3}$$

We can define as $\widehat{f} \equiv \mathcal{F}(f) : \mathbb{R}^n \to \mathbb{C}$ the

$$\lambda \mapsto \widehat{f}(\lambda) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-2\pi i\lambda\cdot x} f(x)\, dx \tag{1.4}$$

and $\widehat{G} \equiv \mathcal{G}(f) : \mathbb{R}^n \to \mathbb{C}$ respectively.

$$x \mapsto \widehat{G}(x) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{2\pi i\lambda\cdot x} f(\lambda)\, d\lambda. \tag{1.5}$$

Theorem 1.0.1. *Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $\widehat{f}$ its Fourier transform. Then, $\widehat{f} \in C^\infty(\mathbb{R}^n)$ and for $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ we have*

$$\frac{\partial \widehat{f}}{\partial \lambda_j}(\lambda) = (-\widehat{i x_j f(x)})(\lambda)$$

and

$$i\lambda_j \widehat{f}(\lambda) = \widehat{\frac{\partial f}{\partial \lambda_j}}, \quad j \in \{1, 2, \ldots, n\}.$$

In general, for $\alpha \in \mathbb{Z}_+$, we have

$$(-1)^{|\alpha|}\partial^\alpha \widehat{f}(\lambda) = (\widehat{i x^\alpha f})(\lambda) \tag{1.6}$$

and

$$(\widehat{\partial^\alpha f})(\lambda) = (i\lambda)^\alpha \widehat{f}(\lambda). \tag{1.7}$$

1.0.2 Differential operators

Let I be a finite index set. Let $X \subset \mathbb{R}^n$ open. Let $(g_\alpha)_{\alpha\in I} \subset C^\infty(X)$ a sequence of functions. We call a Linear Differential Operator with C^∞ coefficients the sum

$$P(x, \partial) = \sum_{\alpha\in I} g_\alpha(x)\partial^\alpha.$$

If $n = 1$ or $n > 1$, then this operator is called *normal* or *partial* respectively. The maximal element m of the set I is called the *order* of P, ie. the element $m := \max\{|\alpha| \ : \ \alpha \in I\}$. Therefore, the operator P takes the form

$$P(x, \partial) = \sum_{|\alpha| \leq m} g_\alpha(x)\partial^\alpha. \tag{1.8}$$

where m is the least number such that there exists (at least) one $g_\alpha \not\equiv 0$, where $\alpha \in I$ with $|\alpha| = m$. Thus, P is a function from the space $\mathcal{D}(X)$ to itself. It is also sequentially continuous. In result, the correspondence $u \mapsto \sum_{|\alpha| \leq m} g_\alpha(x)\partial^\alpha u$ is sequentially continuous and thus for $u \in \mathcal{D}'(X)$, we have that $\sum_{|\alpha| \leq m} g_\alpha(x)\partial^\alpha u$ belongs to the space $\mathcal{D}'(X)$. Specifically, (for $u \in \mathcal{D}'(X)$) Pu is given by the following action

$$\langle Pu, \phi \rangle = \left\langle u, \sum_{|\alpha| \leq m} (-1)^{|\alpha|}\partial^\alpha(g_\alpha \phi) \right\rangle, \quad (\phi \in \mathcal{D}(X)).$$

The operator

$$\phi \mapsto \sum_{|\alpha| \leq m} (-1)^{|\alpha|}\partial^\alpha(g_\alpha \phi)$$

is called the adjoint (operator) of P. Let P be a linear partial operator as in (1.8). We call the operator L_p given by

$$L_p := \sum_{|\alpha| = m} g_\alpha(x)\xi^\alpha$$

as the *principal part* of P. The function

$$P(x, \xi) := \sum_{|\alpha| \leq m} g_\alpha(x)\xi^\alpha : X \times \mathbb{R}^n \to \mathbb{C}$$

is called the *principal symbol* of P. We call

$$\sigma_P(x, \xi) = \sum_{|\alpha| = m} g_\alpha(x)\xi^\alpha$$

the *main symbol* of P. Note that by the Theorem of Leibniz, we have for $f \in C^\infty(X)$ and $u \in \mathcal{D}'(\mathbb{R}^n)$,

$$P(x, \partial)(fu) = \sum_{\alpha \geq 0} \frac{\partial^\alpha u}{\alpha!} P^{(\alpha)}(x, \partial)f,$$

where $P^{(\alpha)}(x, \partial)$ is a linear partial operator with symbol $P^{(\alpha)}(x, \xi) = \partial_\xi^\alpha P(x, \xi)$. We now extend the meaning of a linear differential operator in many dimensions. Let $N > 1$ integer. Consider the function

$$\mathcal{D}(X) \to (\mathcal{D}'(X))^N \quad with \quad \phi \mapsto \langle u, \phi \rangle := (\langle u_1, \phi \rangle, \ldots, \langle u_N, \phi \rangle).$$

Here, $(\mathcal{D}'(X))^N$ stands for $\underbrace{\mathcal{D}'(X) \oplus \ldots \oplus \mathcal{D}'(X)}_{N-times}$. Let

$$P = (P_{k,\ell})_{k,\ell=1}^N = \begin{bmatrix} P_{11} & \ldots & P_{1,\ell} \\ P_{21} & \ldots & P_{2,\ell} \\ \vdots & & \\ P_{k1} & \ldots & P_{k,\ell} \end{bmatrix},$$

where $P_{k,\ell},\ (k,\ell = 1,\dots,N)$ are linear differential operators with C^∞ coefficients. Then, we obtain

$$(\mathcal{D}'(X))^N \to (\mathcal{D}'(X))^N \quad with \quad u := (u_1,\dots,u_N) \mapsto \left(\sum_{\ell=1}^{N} P_{k,\ell}(x,\partial)u_\ell\right)_{k=1,\dots,N}.$$

Here, $u = (u_1,\dots,u_N)$ acts $(\mathcal{D}(X))^N$ as follows:

$$(\mathcal{D}(X))^N \to (\mathcal{D}'(X))^N \quad with \quad \phi := (\phi_1,\dots,\phi_N) \mapsto \langle u,\phi\rangle := \sum_{\ell=1}^{N} \langle u_\ell,\phi_\ell\rangle.$$

Thus, P is defined by

$$\langle Pu,\phi\rangle = \langle u,\ {}^tP\phi\rangle,$$

where ${}^tP$ is the **conjugate** of P, i.e.

$${}^tP = (\ {}^tP_{k,\ell})_{k,\ell=1}^N = \begin{bmatrix} {}^tP_{11} & \dots & {}^tP_{1,\ell} \\ {}^tP_{21} & \dots & {}^tP_{2,\ell} \\ \vdots & & \\ {}^tP_{k1} & \dots & {}^tP_{k,\ell} \end{bmatrix}.$$

Proposition 1.0.1. *Let $P(x,\partial_x)$ a linear differential operator of principal symbol $P(x,\xi)$. Then*

$$(\widehat{P(x,\partial_x)f})(\xi) = P(-\partial_\xi,\xi)\widehat{f}(\xi)$$

and

$$P(\xi,\partial_\xi)\widehat{f}(\xi) = [\widehat{P(\partial_x,-x)f(x)}](\xi).$$

PROOF Follows from linearity and the following identities: for every $\alpha \in \mathbb{Z}_+^n$:

$$(-\partial_\xi)^\alpha e^{-2\pi i x\cdot\xi} = x^\alpha e^{-2\pi i x\cdot\xi} \quad and \quad \partial_x^\alpha e^{2\pi i x\cdot\xi} = \xi^\alpha e^{2\pi i x\cdot\xi}$$

Some notation Let $P(x_1,\dots,x_n)$ be a polynomial if n variables. Then, by $P(\partial) \equiv P(D)$ we denote the linear partial differential operator which occurs under the 'identification' $\frac{\partial}{\partial_i} \equiv \partial x_i \leftrightarrow x_i,\ i = 1,2,\dots,n$ in P, i.e.

$$P(D) = P\left(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_n}\right).$$

For example, if $P(x,y,z) = 9ix^2y + xz^3y + zx$, then

$$P(D) = 9i\frac{\partial^3}{\partial x\partial x\partial z} + \frac{\partial^5}{\partial x\partial z\partial z\partial z\partial y} + \frac{\partial^2}{\partial z\partial x}.$$

With $\mathbb{C}[\partial_1,\dots,\partial_n] \equiv \mathbb{C}[\partial]$ we shall denote the set of all linear partial differential operators.

Definition 1.0.4. *Let P be a polynomial in $\mathbb{R}^n$. If g is a function in $\mathbb{R}^n$, then every $u \in \mathcal{D}'(\mathbb{R}^n)$ which solves the D.E. $P(D)u = g$ is called **a weak solution** of $P(D)$. The same holds for $g \in \mathcal{D}'(\mathbb{R}^n)$. In the case where $g = \delta$, i.e. the Dirac distribution, then every u which is a solution of $P(D)u = \delta$, will be called a **fundamental solution** of $P(D) \in \mathbb{C}[\partial]$.*

Remark 1.0.1. *(i) The zero linear partial differential operator cannot have a fundamental solution. The constant linear partial differential operator $P(D) = c \in \mathbb{C}_*$ has a unique fundamental solution: $v = \frac{1}{c}\delta$. Thus, a fundamental solution cannot exist in L^1_{loc}.*

(ii) If $g \in \mathcal{D}'(\mathbb{R}^n)$, then $P(D)u = g$ means

$$\langle P(D)u, \phi\rangle = \langle g, \phi\rangle$$

for every $\phi \in \mathcal{D}(\mathbb{R}^n)$. If $g = \delta$, then $P(D)u = \delta$ means

$$\langle P(D)u, \phi\rangle = \langle \delta, \phi\rangle = \phi(0)$$

for every $\phi \in \mathcal{D}(\mathbb{R}^n)$.

(iii) If $v \in \mathcal{D}'(\mathbb{R}^n)$ is a fundamental solution for $P(D)$ (i.e. $P(D)v = \delta$), then for every $\phi \in \mathcal{D}'(\mathbb{R}^n)$ we have

$$v * (P(D)\phi) = P(D)(v * \phi) = (P(D)v) * \phi = \delta * \phi = \phi.$$

Thus, if $\phi = \delta$, then

$$v * (P(D)\delta) = \delta$$

therefore v is the inverse convolution of $P(D)\delta$ with support the singleton $\{0\}$. If $v \in \mathcal{S}'(\mathbb{R}^n)$ then the Fourier transform of v is

$$\widehat{v} \cdot P = \widehat{\delta} = 1,$$

i.e. it is the multiplicative inverse of the polynomial $P(\lambda)$.

(iv) If $g \in \mathcal{D}(\mathbb{R}^n)$ and $v \in \mathcal{D}'(\mathbb{R}^n)$ a fundamental solution of $P(D)$, then by setting $u = v * g$, we obtain

$$P(D)u = P(D)(v * g) = (P(D)v) * g = \delta * g = g \tag{1.9}$$

and thus $v * g$ is a solution (under the usual sense).

Another theorem Mckibben uses to solve some Cauchy initial value problems is the Cauchy−Kowalevski theorem and is about the existence of solutions to a system of m differential equations in n dimensions when the coefficients are analytic functions.

Theorem 1.0.2. *If F and f_i are analytic functions near the origin, then the non-linear Cauchy problem*

$$\partial_t^k h = F(x, t, \partial_t^i \partial_x^\alpha h), \quad \textit{where } i < k \textit{ and } |\alpha| \le k - i$$

with initial conditions

$$\partial_t^i h(x, 0) = f_i(x), \ 0 \le i < k$$

has a unique analytic solution near the origin.

This Theorem guarantees the (local) existence and uniqueness for analytic partial differential equations associated with Cauchy initial value problems but is very restrictive, since we require the analyticity of the coefficients.

1.0.3 Principal Values and Finite Parts of Integrals

The function $f_\alpha : \mathbb{R} \to \mathbb{R}$ ($\alpha \in \mathbb{R}$) with $x \mapsto f_\alpha(x) := x^\alpha$ belongs to the space $L^1_{loc}(\mathbb{R})$ for $\alpha > -1$ and it therefore defines a distribution. For $\alpha = -1$, we define as the Cauchy Principal Value Integral and denote with $PV\left(\frac{1}{x}\right)$ the element of $Hom(\mathcal{D}(\mathbb{R}); \mathbb{R})$

$$\phi \mapsto PV\left(\frac{1}{x}\right)(\phi) \equiv:= \lim_{\epsilon \to 0} \int_{\{|x|>\epsilon\}} \frac{\phi(x)}{x} dx.$$

We can easily see that

$$\lim_{\epsilon \to 0} \int_{\{|x|>\epsilon\}} \frac{\phi(x)}{x} dx = \int_{[0,\infty)} \frac{\phi(x) - \phi(-x)}{x} dx = \int_{[0,\infty)} \left(\int_{[-1,1]} \phi'(rx)\, dr \right) dx$$

Theorem 1.0.3. *Let $x_0 > 0$. The map $PV\left(\frac{1}{x-x_0}\right) : \mathcal{D}(\mathbb{R}) \to \mathbb{C}$ defined via the Cauchy Principal Value as*

$$\phi \mapsto PV\left(\frac{1}{x - x_0}\right)(\phi) = \lim_{\varepsilon \to 0^+} \int_{|x-x_0| \geq \epsilon} \frac{\phi(x)}{x - x_0}\, dx$$

*is a distribution called the **Cauchy Principal Value distribution**.*

Definition 1.0.5. *If $f \in C^\infty(\mathbb{R})$ and $x_0 > 0$ such that the Cauchy Principal Value Integral* $\mathrm{PV}\left(\int_a^b \frac{f(x)}{x-x_0}\, dx\right)$ *exists for $x \in [a,b]$, then the Hadamard Finite Part Integral is*

$$\frac{d}{dx}\left(\mathrm{PV}\left[\int_a^b \frac{f(x)}{x - x_0}\, dx\right]\right). \tag{1.10}$$

(1.10) can also be expressed via the ϵ notation as

$$\lim_{\epsilon \to 0^+} \left[\int_a^{x_0 - \epsilon} \frac{f(x)}{(x - x_0)^2}\, dx + \int_{x_0 + \epsilon}^b \frac{f(x)}{(x - x_0)^2}\, dx - \frac{2f(x_0)}{\epsilon} \right].$$

For higher powers of x, the same idea won't work as divergent terms appear under the sign of integration. We overcome this obstacle by discarding those boundary terms that tend to become infinite (the idea is due to Hadamard): we define the (Hadamard) Finite Part of $\frac{1}{x^n}$, ($n \in \mathbb{N},\ n \geq 2$)

$$FP\left(\frac{1}{x^n}\right)(\phi) := \lim_{\epsilon \to 0} \int_{\{|x|>\epsilon\}} \frac{\phi(x) - \sum_{k=0}^{n-2} \phi^{(k)}(0) x^k}{x^n} dx.$$

The above expression is a distribution and in fact one can prove that if $u \in \mathcal{D}'(\mathbb{R})$ such that $x^n u = 1$ (in the sense of distributions), then

$$u = FP\left(\frac{1}{x^n}\right) + \sum_{k=0}^{n-1} c_k \partial_x^{(k)} \delta_0,$$

where $c_k \in \mathbb{C}$ and δ_x is the Dirac Delta function. Specifically, we have the following theorem the division problem in the one dimensional real case:

Theorem 1.0.4. *(i) If $u \in \mathcal{D}'(\mathbb{R})$ and $x^m u = 0$, then*

$$u = \sum_{j=0}^{m-1} c_j \partial^j \delta,$$

where $c_j \in \mathbb{C}$, for $i \in \{1, 2, \dots, m-1\}$.

(ii) If $v \in \mathcal{D}'(\mathbb{R})$ is given, then there exists a $u \in \mathcal{D}'(\mathbb{R})$ such that

$$x^m u = v.$$

Hence, if w is any solution of the problem $x^m u = v$, then the general solution of the problem is of the form

$$u = w + \sum_{j=0}^{m-1} c_j \partial^j \delta.$$

In order to prove this, let $\phi \in \mathcal{S}(\mathbb{R})$ and P as above such that ϕ does not contain any other zeroes of P. The Taylor expansion of ϕ around $x = x_0$ of order $m = \min\{n, a : n \in \mathbb{N},\ n > a - 1\}$ is

$$\phi(x) = \sum_{j=0}^{m-1} \frac{\phi^{(j)}(x_0)}{j!}(x - x_0)^j + R_m(x),$$

where R_m is the remainder of the expansion. Then, it is easy to prove that the correspondence

$$\langle \frac{1}{(x - x_0)^a}, \phi \rangle := \int_{x_0}^{\infty} \frac{R_m(x)}{(x - x_0)^a}\, dx \tag{1.11}$$

is our distribution.
The study of the division problem in dimensions $n \geq 2$, using the method of Mckibben, requires the following formula, which is essentially Green's formula on Linear differential operators:

Theorem 1.0.5. *Let (M, g) be a compact oriented Riemannian manifold with boundary. Let E, F be two Hermitian bundles over M and let $P : S(E) \to S(F)$ be a linear differential operator of order m. Then, there is a sesquilinear map $G^P : S(E) \times S(F) \to C^\infty(M; \mathbb{C})$ such that for every pair of sections $(u, v) \in S(E) \times S(F)$, the following formula holds*

$$\int_M \langle Pu, v \rangle = \int_M \langle u, P^* v \rangle + \int_{\partial M} G^P(u, v).$$

Here, we only need the following version of the above theorem:

Theorem 1.0.6. *Let $u, v \in C^m(\mathbb{R}^n)$ (where m is a positive integer) and let $U \subseteq \mathbb{R}^n$ with sufficiently smooth boundary. Let P a linear differential operator of order m. Then*

$$\int_U vPu dx_1 \dots dx_n = \int_U uP^* v dx_1 \dots dx_n + \int_{\partial U} G(u, v) d\sigma,$$

where $d\sigma$ is the volume element of the boundary of U and G is a function on the boundary of U such that its value on and point of the boundary depends only on the direction of the normal vector to the surface and on the values of u, v and their derivatives of order $\leq m - 1$.

Both Theorems represent integration by parts. Note also, that both Theorems still hold if we replace the adjoint operator with its complex conjugate one.

Definition 1.0.6 (Characteristic hypersurface)**.** *Let S be the hypersurface in $\mathbb{R}^n$ defined by the equation $s(x) = 0$, where s is a smooth function. Suppose that $\nabla s(x) \neq 0$, for each $x \in S$. Let P a linear differential operator of order m. If $P_m(x, \nabla s(x)) = 0 \; \forall x \in S$, then S is called a characteristic surface or a characteristic for the operator P. In the opposite case, S is called non-characteristic.*

Since at some point of this thesis we shall stumble upon some Cauchy initial value problems, we outline the relationship between characteristic surface. Let S be the hypersurface in $\mathbb{R}^n$ defined by the equation $s(x) = 0$, where s is a smooth function. Let fi, $i = 1, 2, \ldots, m$ be sufficiently smooth functions defined in a sufficiently small neighborhood U of a point $x_0 \in S$. Let P a linear differential operator of order m. The Cauchy problem for the unknown function f_1 is

$$\begin{cases} P(x, \partial_x)f = g & (x \in U) \\ \\ f = f_1, \; \frac{\partial_n^{i-1} f}{\partial n^{i-1}} = f_{i-1}, \; for \; i = 2, 3, \ldots, m & (x \in S) \end{cases}$$

where g is a given function, and n is an orthonormal to the hypersurface S vector. Assume without loss of generality that $\frac{\partial s(x)}{\partial x_n} \neq 0, \; \forall x \in U$. Then by performing the change of variables

$$y_i = x_i, \quad i = 2, 3, \ldots, n-1, \; y_n = s(x),$$

$P(x, \partial_x)f = g$ $(x \in U)$ is equivalent to

$$P_m(x, \nabla s(x)) \frac{\partial^m f(x)}{\partial y^m} + terms = g, \tag{1.12}$$

where 'terms' do not involve partial derivatives of f of order m (with respect to the variable y_n). If the hypersurface S is characteristic, then the above Cauchy problem becomes very complicated. In the opposite case, i.e. if $P_m(x, \nabla s(x)) \neq 0$, $\forall x \in U$, then we can divide (1.12) by $P_m(x, \nabla s(x))$ and obtain the equivalent initial conditions

$$\frac{\partial^i f(y_1, \ldots, y_{n-1}, 0)}{\partial y_n^i} = f_i(y_1, \ldots, y_{n-1}), \quad i = 1, 2, \ldots, m.$$

Furthermore, if the coefficients of P, function f and the vector function $(f_1, \ldots, f_n)$ are real analytic, Theorem (1.1.2), guarantees the existence of a unique solution of this problem in the class of real-analytic functions in a sufficiently small neighbourhood of a point x_o of U. For more, see [20].

Definition 1.0.7. *Let $U \subseteq \mathbb{R}^n$ open. Let $P(x, \partial_x)$ a partial diff. operator of order m. Given a point $x_o \in U$, the vectors ξ satisfying*

$$P_m(x_0, \xi) = 0, \tag{1.13}$$

are called the ***characteristic vectors*** *of P at x_0. The hypersurface (in the $\xi-$plane) consisting all characteristic vectors of P at x_0 is called the* ***characteristic cone*** *at x_0.*

The characterization 'cone' stems from the fact that P_m is homogeneous (of degree m) and thus (1.13) remains invariant under rescaling by a complex number.

Definition 1.0.8. *Let $P(x, \partial_x)$ a partial diff. operator acting on the space $C^\infty(U)$, where $U \subseteq \mathbb{R}^n$ open. Then, $P(x, \partial_x)$ is called* ***elliptic*** *if*

$$P_m(x_0, \xi) \neq 0, \; \forall \xi \neq 0, \; \forall \in x \in U. \tag{1.14}$$

It is clear that the characteristic cone at any point x_0 of an elliptic operator degenerates to a single point, i.e. $\xi = 0$.